Qualitative analysis of systems of ODEs

Recall that the qualitative analysis of dynamical systems has in general three steps:

1. Identify equilibria (fixed points) of the system. This includes the conditions (mathematical and biological) for existence of each equilibrium. In case the equilibrium conditions are too complicated to solve outright, one may nonetheless be able to determine conditions on the existence (and number) of equilibria.

2. Determine conditions for the local asymptotic stability of each equilibrium, by linearizing the system about each equilibrium in turn. For ODE systems this process typically involves ensuring that the eigenvalues of the corresponding Jacobian matrix have negative real part. In cases where the eigenvalues cannot be computed directly, one can use the Routh-Hurwitz criteria to determine stability conditions.

The Routh-Hurwitz criteria for a two-dimensional system are that $a_1 > 0$ and $a_2 > 0$ for the coefficients of the Jacobian matrix’s characteristic equation $\lambda^2 + a_1 \lambda + a_2 = 0$; in terms of the matrix $J$ these criteria can be rewritten as trace $J < 0$ and det $J > 0$. (If det $J < 0$ then the equilibrium is a saddle point (and thus unstable), while if det $J > 0$ but trace $J > 0$ then it is an unstable node or spiral. Recall that if either quantity is exactly zero, which corresponds to the real part of at least one eigenvalue of $J$ also being zero, then the linearized system gives no information about the stability of the equilibrium in the nonlinear system.)

The Routh-Hurwitz criteria for a three-dimensional system are that $a_1 > 0$, $a_3 > 0$, and $a_1 a_2 > a_3$ for the coefficients of the Jacobian matrix’s characteristic equation $\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0$.

The Routh-Hurwitz criteria for systems of dimension higher than three can be found in numerous references, but normally lead only to great sadness, so I will not give them here.

3. Compose a global behavior portrait: that is, describe qualitatively the behavior of all possible solutions to the system. There are several specialized tools to aid in this work, notably:

- Lyapunov functions for proving the global asymptotic stability of equilibria,
- phase portraits for assembling local information into a global portrait,
- *ad hoc* techniques like bounding boxes for proving that no solutions grow unbounded,
- the Poincaré-Bendixson Theorem for classifying behavior of solutions to two-dimensional systems,
- Bendixson’s Criterion, Dulac’s Criterion, and applications thereof, to rule out periodic orbits in two-dimensional systems,
- bifurcation analysis,
- assorted techniques for reducing the effective dimension of a system.

I leave further review of local stability analysis and phase portraits to various bibliographic references (q.v.). We now consider Lyapunov functions; the remaining tools will be explained later.
Lyapunov functions

A strong Lyapunov function is a function \( V \in C^1 \) on the state space of a system \( d\vec{x}/dt = f(\vec{x}) \) with equilibrium \( x^* \) such that:

(i) \( V(x^*) = 0 \), and \( V(x) > 0 \) for all \( x \neq x^* \) (\( V \) is said to be positive definite);
(ii) \( \frac{dV}{dt}(x^*) = 0 \), and \( \frac{dV}{dt}(x) < 0 \) for all \( x \neq x^* \) (\( \frac{dV}{dt} \) is said to be negative definite).

If a strong Lyapunov function exists for a given system and equilibrium, then that equilibrium is globally asymptotically stable (GAS).

A weak Lyapunov function is positive definite, but its time-derivative \( dV/dt \) is only negative semidefinite: that is, \( \frac{dV}{dt}(x^*) = 0 \), and \( \frac{dV}{dt}(x) \leq 0 \) for all \( x \neq x^* \). In this case, there is an additional condition required to prove global stability of \( x^* \), provided by LaSalle’s Invariant Principle:

Given a weak Lyapunov function \( V(x) \) for a system \( dx/dt = f(x) \) with equilibrium \( x^* \), then \( x^* \) is GAS if \( x^* \) is the only invariant subset of the set \( \{ x : dV/dt(x) = 0 \} \). (More generally, the largest invariant subset of this set is globally attracting.)

In other words, we must consider the set of points where \( dV/dt = 0 \) and verify that, if at some time \( t \) a solution reaches such a point \( x \neq x^* \), the solution does not then remain in the set \( \{ x : dV/dt(x) = 0 \} \); rather, it leaves the set immediately, so that \( V(x(t)) \) continues to decrease. We can verify that this property holds by substituting into the ODE system the conditions for the set where \( dV/dt = 0 \) and seeing that the vector flow \( f(x) \) then sends the solution outside that set (unless \( x = x^* \)).

These definitions explain why it can be useful to have a Lyapunov function for a given system and equilibrium, but how can we find one? In general there is no single methodical approach to finding one (indeed there are many systems with equilibria which are LAS but not GAS, and thus no Lyapunov functions can exist for those equilibria). However, it is sometimes possible to create one by taking a linear combination of the state variables, with coefficients chosen in such a way to simplify the time-derivative \( dV/dt \) to a point where it can be proven negative; in particular, if \( dx_i/dt \) contains a term of form \( g \) and \( dx_j/dt \) \((j \neq i)\) contains a term of form \(-kg \) \((k \) constant), then the combination \( kx_i + x_j \) has a time-derivative in which the two corresponding terms cancel. Note that one may also add a constant to the candidate for \( V \), in order to make \( \frac{dV}{dt}(x^*) = 0 \), without disturbing \( dV/dt \) at all; in fact all that is really important for condition (i) is that \( x^* \) be the unique global minimum of \( V \).

Another strategy which is sometimes helpful is to try a function of the form \( V = \sum_{i=1}^{n} a_i x_i^2 \) where the state variables are \( x_1, x_2, ..., x_n \) (translated if necessary to move the equilibrium to the origin) and the \( a_i \) are suitably chosen (nonnegative) constants. A related strategy (see the book by James R. Leigh, Sections 7.6–7.7) is to add to the above sum of squares one or more cross terms of the form \( bx_ix_j \) with coefficients small enough that one can still prove \( V \) is positive definite; for example, in a 2-dimensional system with variables \( x \) and \( y \), try \( V = ax^2 + hxy + by^2 \) with \( h^2 < ab \).

A very simple example of a Lyapunov function is, for the single equation \( dx/dt = -x^3 - x^5 \), which has an equilibrium \( x^* = 0 \), the function \( V = x^2 \), which is clearly positive definite, and for which

\[
\frac{dV}{dt} = 2x \frac{dx}{dt} = 2x(-x^3 - x^5) = -2x^4 - 2x^6,
\]

which is negative for all \( x \neq 0 \). Thus (since a Lyapunov function exists for \( x^* = 0 \)) the equilibrium is GAS.