Mathematics 1330

Arithmetical
Problem Solving

University of Texas at Arlington

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Math 1330
UNIT ONE
Problem Solving

The problems in this unit require a variety of different kinds of mathematical approaches, and were selected to give you some experience with the different mathematical situations in which you can find yourself. It may help you to keep a list of the different problem solving techniques that you use on the problems you solve, so that if you get stuck later on, you can come back to your list for some ideas.

POISON

A Game

Number of Players: Two teams of one or more players

Equipment: Counters

Rules:

1. Begin by putting 10 counters on the table between you. Designate one of them as the "poisoned" counter.
2. The first time you play, toss a coin to decide which team goes first; after that, alternate which team begins the game.
3. On your team's turn, you *must* take away either 1 or 2 counters.
4. The team forced to take the last (POISONED) counter loses.

Instructions:

1. Play the game a few times. As you play, consider these questions: how many counters did you pick up? *Why?* Who won the game?
2. After you have played the game a few times, reflect on your game records and see if your group can devise a winning strategy. Once you think you have one, test it out (against other players).
3. Once you're sure you know the winning strategy, try starting with a different number of counters than 10. How does the winning strategy change?
4. Suppose you had to play a game of Poison using 431 counters. Would you care whether you went first or second? Why?

Notes and Game Records:

Variations on a Theme

One way to see how thoroughly you understand the solution to one problem is to try to apply it to similar or related problems. Below are three variations on the game Poison. Throughout this coursepack there are pages like this intended for practice and application outside of class, with which you can test yourself to see how solid your understanding is of the basic principles.

Each of the games below has just one difference from Poison in its rules; as a consequence, the winning strategy has the same general structure as that for Poison. However, don't just jump to a conclusion and guess what aspect of the winning strategy has changed — try it out with a handful of counters, and observe the pattern that emerges from the endgame, just as you did in analyzing Poison (but ideally faster, now that you get the hang of it).

1. **Super Poison** — In Super Poison, you may take 1, 2, or 3 counters away on your turn. The team that takes the last counter is still the loser.

2. **Antidote** — Were you forced to take a poisoned counter while playing Poison? Don't worry, now you have a chance to take the antidote!

   In this game, you may take away 1 or 2 counters on your turn, as in Poison, but now whoever takes the last counter (the "antidote") is the *winner*.

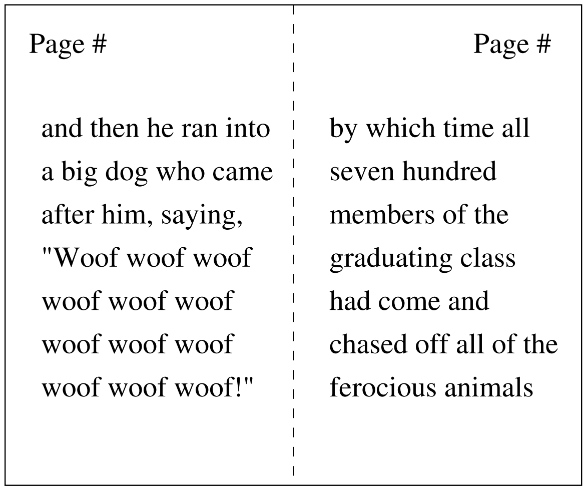
3. **Rat Poison** — *to win, you have to be as sneaky as a rat!*

   RAT POISON has the same rules as POISON, except that in RAT POISON you may take 1, 3, or 4 counters on your turn; you *cannot* take 2 counters. The team that takes the last counter is still the loser.

BOOKS and PAGES

This problem is adapted from C. Greenes, R. Spungin, & J. Dombrowski, *Problem-mathics*, Palo Alto, CA: Creative Publications, Inc. (1977).

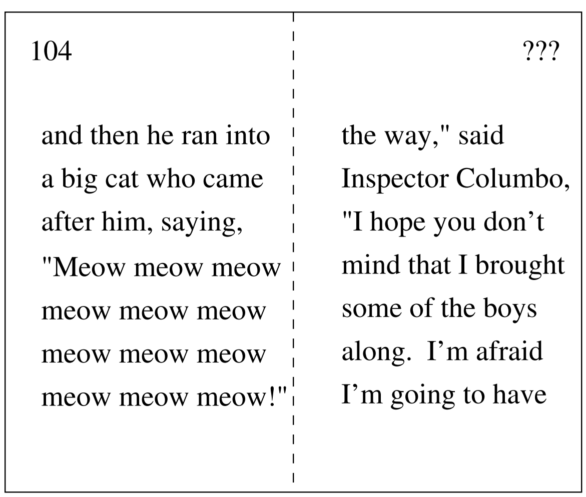
Folios. Books are sometimes made up of sections of pages called folios. In a folio each sheet of paper is printed as shown on both the front and the back.



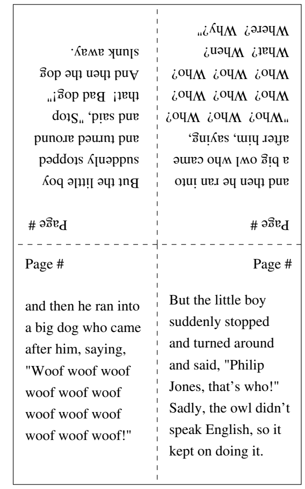
After printing, the folio pages are stacked up, folded (in half) along the dashed line shown above, and sewn together on that fold with nylon cord.

1. A short novel, *The Golden Arch*, has 156 pages. How many folio sheets were needed? How did you figure this out?

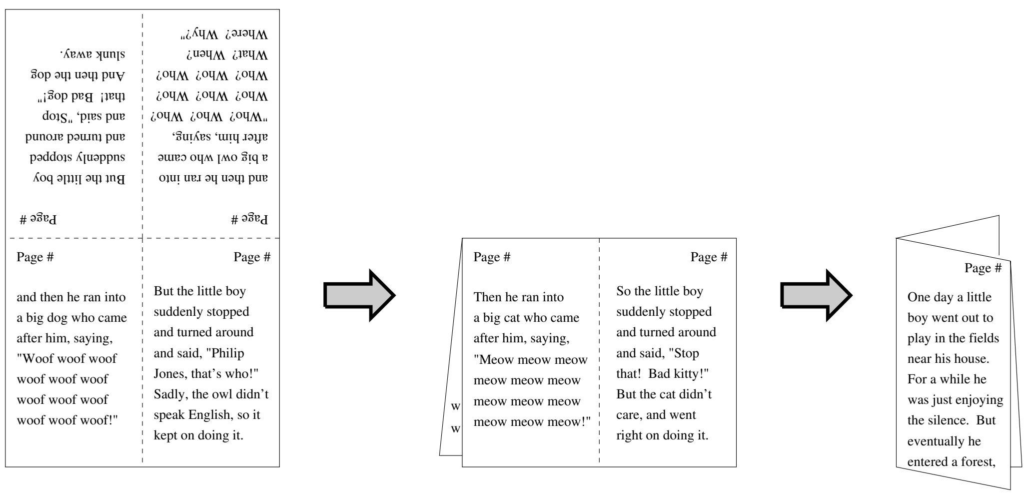
2. The folio page shown below (from *The Golden Arch*) fell on the floor. What is the number on the other corner (that is, where ??? appears)? What are the two numbers on the other side of this sheet of paper?



Quartos. In quarto publishing, four book pages are printed on each side of the sheets of paper used. One side of a quarto sheet is shown at right.



Once the sheets are printed on both sides, they are stacked up, and the stack is folded twice: first from top to bottom on the dashed horizontal line and then from side to side on the dashed vertical line. The tops of the pages are cut apart. Thus, every quarto sheet produces eight book pages.



1. How many quarto sheets are needed for a book of 128 pages?

2. When the pages of this book are stacked and ready to be folded, what does the top sheet look like?

3. Draw a picture that shows where on the sheet the page numbers appear and which are upside down.

4. What other page numbers are on the the same quarto sheet with page 13?

Time to Weigh the Hippos

Martha is the chief hippopotamus caretaker at the Wild Animal Park in San Diego, California. She has just arrived at the cargo dock in order to pick up four members of the endangered species *hippopotamus mathematicus* that were recently rescued from African poachers. Before the animals are released by the harbormaster, Martha must weigh them. BUT the only scale big enough to weigh a hippo is the truck scale that doesn t weigh anything lighter than 300 kilograms (kg); this is more than each of the hippos weighs. Martha is puzzled for a few minutes, then gets the idea of weighing the hippos in pairs, thinking that if she gets the mass of every possible pair, she can later figure out the masses of the individual hippos. She weighs the hippos pair-by-pair and gets 312 kg, 356 kg, 378 kg, 444 kg, and 466 kg (not necessarily in that order). When she tries to weigh the heaviest pair of hippos, the scale breaks. Alas!

1. What was the mass of the last pair of hippos who broke the scale?

2. What are the masses of the individual hippos?

NOTE: Write down explicitly any assumptions you make, and explain your reasoning at each step.

Utopia

The Utopia ISD administration building is located near the center of Utopia, at the intersection of Main Street and Broadway Avenue. Utopia has four elementary schools. Two are located on Broadway Avenue, one on the north side of town and one on the south side. The other two are located on Main Street, one on the east side of town and the other on the west side. The Superintendent has to fill out a new form that states how far (to the nearest tenth of a mile) each elementary school is from the administration building.

To save time and gasoline, the Superintendent wants to find the distances without actually sending someone out to measure them. She thinks the music teacher might know the distances because the music teacher has to visit two different elementary schools each day. The music teacher tells the Superintendent that he always drives directly from one school to another, passing through the intersection of Main and Broadway, and that five of these school-to-school distances add to 7.6 miles, 7.8 miles, 9.4 miles, 9.6 miles, and 12.0 miles, but he can't give a figure for the distance between the two schools closest to the administration building, because he always takes a shortcut when he travels between those two schools. (The music teacher is always thinking about driving safely and how to endure those dreadful kazoos at the next stop, so he never notices how far any particular school is from the administration building.)

The Superintendent hopes to figure out the information she needs for the form from this data. She is puzzling over this problem when you arrive for your job interview, and asks you to help her.

What are the individual distances of the four elementary schools from the administration building? (Also state a simple thing the Superintendent can do to resolve the dilemma you will inevitably discover when you work on this problem.)

NOTE: Write down explicitly any assumptions you make, and explain your reasoning at each step.

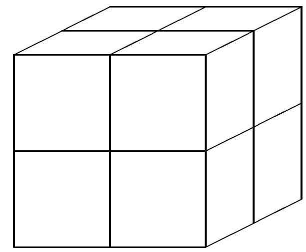
A Frame-Up

I painted a rectangular picture that is 4 inches longer than it is wide. If I put a 3-inch frame around my picture, the area increases by 48 square inches. What are the dimensions of my picture?

Four Cuts

If you were to take a block of cheese and make a straight (i.e., planar) cut all the way through, you would end up with two pieces. If you were then to make another cut without moving either of those pieces, you should be able to do it in such a way that you end up with four pieces. What is the maximum number of pieces you can have after three cuts (remember you can't move any of the pieces)? What about after four cuts? You should be ready to explain how to make the cuts so as to obtain the maximum possible number of pieces.

Painting the Cube



Form a $2 \times 2 \times 2$ cube by taking eight smaller cubes and stacking them together. Now imagine that, while holding the small cubes together in the shape of the larger cube, you paint all the exposed faces. If you take the big cube apart after the paint dries and look at the individual smaller cubes, will any of them have *no* paint at all on them? Will any of the small cubes have paint on only 1 face? on 2 faces? 3 faces? more than 3 faces?

Now do the same thing for a $3 \times 3 \times 3$ cube (how many small cubes will you need to make it?). If you painted the exposed faces of the 3-cube, how many of the smaller cubes will have paint on 0 faces? on 1 face? 2? 3?

Fill in the table below with the numbers of each type of cube, and repeat for a $4 \times 4 \times 4$ cube. Continue until you see a pattern which you can describe in terms of the size of the larger cube. In other words, your ultimate goal (the last line of the table) is to say how many of each type of small cube there would be in a large $n \times n \times n$ cube.

# small cubes on 1 edge of large cube	# small cubes with paint on NO faces	# small cubes with paint on 1 face	# small cubes with paint on 2 faces	# small cubes with paint on 3 faces	TOTAL # small cubes in large cube
2					
3					
4					
n					

Can you prove using algebra that the numbers of cubes with paint on 0, 1, 2, or 3 faces *always* adds up to the total number of cubes needed, for an $n \times n \times n$ cube?

Painting the Cube II

This problem is a sequel to Painting the Cube. If you try it, I suggest using physical models as done with the original problem.

Suppose you have 27 identical small cubes, which you can put together to create a single large 3x3x3 cube. Suppose also that you have 3 different colors of paint (for example, red, yellow and blue). Find a way to paint each face of each small cube with one of the three colors, in such a way that:

- there is a way to arrange and assemble the small cubes into the large cube so that only red faces are showing (the large cube appears all red);
- there is a way to arrange and assemble the small cubes into the large cube so that only yellow faces are showing; AND
- there is a way to arrange and assemble the small cubes into the large cube so that only blue faces are showing.

A solution to this problem consists of a description of each of the 27 cubes which clearly tells what it looks like — in other words, which would enable someone who has the unpainted small cubes, 3 cans of paint and 3 paint brushes to actually construct your solution. Be sure to note when faces are adjacent (having two opposite faces painted yellow is not the same as having two adjacent faces painted yellow). You do NOT, however, have to tell how to assemble the small cubes in each of the 3 different ways. You can if you want, but don't get bogged down in that. You should tell how you figured out the answer, and (hint hint) you should start with the corresponding part of the solution to the first Painting the Cube problem (e.g., how many cubes will need red paint on 3 faces? etc.).

The much simpler solution for the 2x2x2 case is done here as an example:

> Eight small cubes form one large 2x2x2 cube. We recall from the first problem that all eight cubes would have paint (of a given color) on 3 faces. Therefore, to paint the cubes with 2 colors, each cube needs 3 faces painted red and 3 faces painted yellow. If you take each small cube and paint one corner (three faces which touch at a single vertex) red, and the opposite corner (the three other faces, which touch at the opposite vertex) yellow, then you can put them together in such a way that only red faces (corners) show, or a different way such that only yellow faces show.

Note that the solution for the 3x3x3 case will NOT have all 27 cubes painted identically — there will be differences, as reflected in the data for the first problem.

Inverse Functions & the Painted Cube

1. Recall the "Painting the Cube" problem. Suppose I found a big pile of painted unit cubes which someone had used to model this problem. If I went through and collected all the cubes with paint on just one face, and counted 24 of them, then how many cubes in that pile must have had paint on exactly 2 faces? 0 faces? Write down how you know.

2. Now suppose instead that I had counted 600 cubes with paint on one face. How many cubes in the pile must have had paint on exactly 2 faces? 0 faces? Write down how you know, including all the computations you made (even the ones you made in your head).

3. In general, if there are x cubes with paint on one face, and y cubes with paint on 2 faces, what is the relationship between x and y? Write down an equation or formula that relates x and y without using any other variables (like n).

4. Rewrite the formula you developed above (a) to solve for y, so that you could use it to find y if you know x, and (b) to solve for x, so that you could use it to find x if you know y.

   (a) $y =$ (b) $x =$

5. In general, if there are x cubes with paint on one face, and z cubes with paint on 0 faces, what is the relationship between x and z? Write down an equation or formula that relates x and z without using any other variables (like n).

Chickens & Rabbits

It is important to be able to explain something in several different ways. See how many *genuinely* different ways you can find to solve the following problem (e.g., numeric, algebraic, graphical, ...).

> In the barnyard there are some chickens and some rabbits. There are 50 heads and 120 legs on these chickens and rabbits. How many chickens and how many rabbits are there in the barnyard?

Make a detailed correspondence among your representations: for instance, how is the number of heads manifested in each representation?

Group Dynamics

1. What qualities of an individual are desirable or necessary to function well within a group (large or small)?

2. What collective qualities of a small group are desirable or necessary in order to function well, both by itself and in a large-group context?

3. What specialized roles may individuals take when working within a group (small or large)?

Anatomy of a Problem

Review your work on the major problems of this unit, and try to identify the distinct phases you went through while solving each. For example, what was the first thing you tried on each problem? What event or realization made you stop doing that and start doing something else instead?

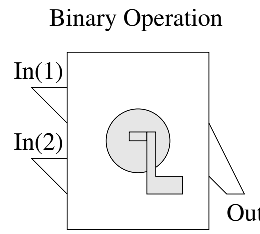
After you have identified the "anatomy" of each problem individually, put your analyses side by side and try to identify any common features. Can you say anything about the general anatomy of a math problem?

Math 1330
UNIT TWO
Operations

The questions and problems in this section are designed to help you see the underlying structures of the four arithmetic operations, which you use every day and probably give no further thought. Other, less familiar examples of operations are provided to give you a basis for comparison.

Introduction to Binary Operations

An *operation* is a way to combine some elements of a set to get another element of the set. You can think of an operation like a machine, with a certain number of slots for inputs and a slot for the output. You put the inputs into the appropriate slots, turn a crank, and out comes the output. You are already familiar with many operations: addition, subtraction, multiplication and division are all operations. These four operations each take two numbers as input, and give one number as output. In general an operation (which you can also think of as a *function*) could take any number of things as input and give any number of things as output. Think, for instance, of a very small scale, which takes as input any single small object and gives as output its weight (a number plus a unit).



The four arithmetic operations are called *binary operations* because they take two inputs. In this course we will consider mostly binary operations.

1. There are many different ways to *represent* an operation.

(a) Represent the operation of addition using: words; a table; symbolically; and with a concrete model (sketch it below).

(b) How does each representation communicate something different?

2. Operations can be defined on any kind of set, not just numbers. Let's consider the following set S: $\{\Box, \bigcirc, \triangle, \star, \heartsuit\}$. Now let's define an operation $\bowtie$ on S by making the output of $a \bowtie b$ equal to whichever of the symbols a and b in S has more (sharp) corners. Thus $\Box \bowtie \bigcirc = \Box$ since $\Box$ has four sharp corners while $\bigcirc$ has none.

Another way to represent a binary operation on a set of finite size is to make a table showing all the possible outcomes. The elements of S run down the left side and across the top. The fact that $\Box \bowtie \bigcirc = \Box$ has been filled in on the table. Complete the table by filling in the rest of the spaces.

$\bowtie$	$\bigcirc$	$\heartsuit$	$\triangle$	$\Box$	$\star$
$\bigcirc$					
$\heartsuit$					
$\triangle$					
$\Box$	$\Box$				
$\star$					

3. Here are some examples of inputs and outputs for some binary operations. Identify which operation is being performed inside the box.

	Inputs	Output
(a)	3, 2	5
	1, 6	7
(b)	3, 2	3
	1, 6	6
(c)	3, 2	3
	1, 6	1

	Inputs	Output
(d)	$+$, O	$\oplus$
	$\times$, |	$*$
(e)	cat, dog	mammal
	cat, fish	vertebrate
	cat, ant	animal

4. Now see if you can work backward to figure out the algebraic rule for the operation $\odot$ defined on the set $\{0, 1, 2, 3, 4\}$ by the table at right.

$\odot$	0	1	2	3	4
0	0	-2	-4	-6	-8
1	3	1	-1	-3	-5
2	6	4	2	0	-2
3	9	7	5	3	1
4	12	10	8	6	4

Properties of Operations

Each operation has a particular underlying structure that can be characterized through *properties*. For example, if I am operating on two numbers, does it matter in which order the two numbers are given? Is the answer also always a number? Is there a particular number that will make the answer always come out to be the *other* of the two numbers? Of course, the answer depends on the operation, and on the set over which it is defined. (For example, the answer to the last question above for addition is yes only if we are including 0 in the set of numbers we can add.)

Mathematicians have identified six of the most important properties operations can have, and given them the name *field properties*. A set of numbers, together with two operations (usually thought of as addition and multiplication, although there are others that will work as well), is called a *field* if the two operations satisfy the six properties: closure, commutativity, associativity, distributivity, identity, and inverses. Let us consider the four familiar arithmetic operations (+, −, ×, ÷), and the two operations GCF (Greatest Common Factor) and LCM (Least Common Multiple), in this context as we define each property.

1. CLOSURE: When any two numbers from the given set are operated on, is the result always a number from this set?

   For GCF and LCM defined on the counting numbers, or natural numbers, 1,2,3,..., the answer is yes: the GCF or LCM of two counting numbers is always a counting number, too.

   (a) Which of +, −, ×, ÷ also have closure on the *natural* numbers?

   (b) Are there any that do not have closure on the natural numbers, but have closure on some other set of numbers?

   (c) What visual clue is there in a table that lets you know whether a set is closed under a given operation? (Hint: Try making tables for the operations you indicated above did not have closure.)

2. COMMUTATIVITY: When any two numbers from the given set are operated on, can you switch the order of the numbers without changing the result?

   For the GCF and LCM defined on the counting numbers, the answer is yes: the GCF of two numbers is the same, regardless of the order in which you list them. Likewise for the LCM.

   (a) Which of +, −, ×, ÷ are also commutative on the natural numbers?

   (b) Are there any that are not commutative on the natural numbers, but are commutative on some other set of numbers?

   (c) What visual clue is there in a table that lets you know whether a given operation is commutative? (Hint: Try making tables for the operations you indicated above are commutative.)

3. ASSOCIATIVITY: When you are performing the operation twice, on a total of three numbers, can you move the parentheses from $(a * b) * c$ to $a * (b * c)$ without changing the result?

For the GCF and LCM defined on the counting numbers, the answer is yes: calculating $GCF(GCF(a, b), c)$ gives the joint GCF of all three numbers, as does $GCF(a, GCF(b, c))$, and likewise for LCM.

(a) Which of $+$, $-$, $\times$, $\div$ are also associative on the natural numbers?

(b) Are there any that are not associative on the natural numbers, but are associative on some other set of numbers?

(c) Why would it be difficult to find a clue to associativity or distributivity in a table?

4. DISTRIBUTIVITY: When you are multiplying a sum by another number, do you get the same answer as if you multiplied each of the original addends by the other factor and then added?

This property is different than the others, in that it involves two different operations working together, in an asymmetric way. The (true) claim that "multiplication distributes over addition" is usually written abstractly as "$(a + b) \times c = (a \times c) + (b \times c)$, and $a \times (b + c) = (a \times b) + (a \times c)$". We will not consider GCF and LCM distributing over each other, but will instead ask: over which other operation(s) does multiplication also distribute? (In other words, with what operation can you replace $+$ in the equation above, and it still hold true?)

5. IDENTITY: Is there a number in the set such that operating on that number and another number always makes the answer the other number (regardless of order)?

   For the LCM defined on the counting numbers, the answer is yes, the number 1: taking $LCM(1, x)$ or $LCM(x, 1)$ always gives a result of x, regardless of which number x is. For this reason, the number 1 is called the identity element for LCM. The GCF operation, however, has no identity element—it would have to be a number Q such that $GCF(Q, 2) = GCF(2, Q) = 2$, $GCF(Q, 3) = GCF(3, Q) = 3$, etc., which would imply that Q has 2 as a factor, and 3 as a factor, etc., but there is no number which has every number as a factor.

   (a) Which of $+$, $-$, $\times$, $\div$ have an identity element in the natural numbers?

   (b) Are there any that have no identity element in the natural numbers, but have an identity element in some other set of numbers?

   (c) What is the visual clue in a table that lets you know whether an operation has an identity element?

6. INVERSES: Does every number in the set have an "opposite number" so that operating on the two of them together gives the identity element as the answer?

   Since inverses are defined in terms of the identity element, we can only ask the question of operations with identities (so GCF is ruled out). The LCM does have an identity element, but does not have inverses, since there is no way to "undo" the LCM operation. The LCM-inverse of the number 7, for instance, would be a number n such that $LCM(n, 7) = LCM(7, n) = 1$—but 1 is not a multiple of 7.

   (a) Which of $+$, $-$, $\times$, $\div$ have not only an identity element, but also inverses as well? Under what number system?

   (b) What visual clue in a table indicates whether an operation has inverses?

7. The mathematical definition of a field does allow one exception to the six properties above: in order to be a field, it is not necessary for the additive identity element to have an inverse under multiplication. Does this matter? Under which number system(s) do normal addition and multiplication therefore constitute a field?

Examples of Operations

In all examples, the operation is defined on the set $S = \{1, 2, 3, 4\}$.
Try to explain why each operation has (or does not have) the given properties.

$\sim$	1	2	3	4
1	1	2	3	4
2	2	3	4	5
3	3	4	5	6
4	4	5	6	7

1. The operation $\sim$ is **not** closed. It is commutative and associative.

The identity is 1.

We see that $inv(1) = 1$, but 2, 3 and 4 do not have inverses.
Thus S does not have inverses with respect to $\sim$.

Δ	1	2	3	4
1	3	4	1	2
2	4	1	2	3
3	1	2	3	4
4	2	3	4	1

2. The operation Δ is closed, commutative and associative.

The identity is 3.

We see that $inv(1) = 1$, $inv(2) = 4$, $inv(3) = 3$ and $inv(4) = 2$.
Thus S does have inverses with respect to Δ.

$*$	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

3. The operation $*$ is closed, commutative, and associative.

The identity is 2.

We see that $inv(1) = 1$, $inv(2) = 2$, $inv(3) = 3$, $inv(4) = 4$,
so all elements of S have inverses with respect to $*$.
Thus we say that $*$ has inverses on S.

$\heartsuit$	1	2	3	4
1	1	2	3	4
2	2	4	2	4
3	3	2	1	4
4	4	4	4	4

4. The operation $\heartsuit$ is closed, commutative, and associative.

The identity is 1.

We see that $inv(1) = 1$, $inv(3) = 3$, but 2 and 4 do not have inverses,
so we say that $\heartsuit$ does not have inverses on S.

$\circ$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	2	3	1

5. Now it's your turn to decide which properties $\circ$ has on S.

Analyzing Operations

Try to see which of the five properties defined on the Properties page the operations listed below have. If there is an identity or inverse elements, say what they are.

1. The four arithmetic operations $+$, $-$, $\times$ and $\div$, on the set of integers $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

2. The operation $\smile$ defined on the set $\{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$ by the table below:

$\smile$	$\heartsuit$	$\diamondsuit$	$\spadesuit$	$\clubsuit$
$\heartsuit$	$\heartsuit$	$\diamondsuit$	$\spadesuit$	$\clubsuit$
$\diamondsuit$	$\diamondsuit$	$\spadesuit$	$\clubsuit$	$\heartsuit$
$\spadesuit$	$\spadesuit$	$\clubsuit$	$\heartsuit$	$\diamondsuit$
$\clubsuit$	$\clubsuit$	$\heartsuit$	$\diamondsuit$	$\spadesuit$

3. The exponentiation operation $\wedge$ defined on the natural numbers $\{1, 2, 3, ...\}$ by $a \wedge b = a^b$.

4. The operation # defined on the natural numbers by $a\#b = ab + 1$.

Word Problem Classification

Addition and Subtraction

Problem Type

Join	**Separate**	**Compare**	**Part-Part-Whole**
Result Unknown Tom had 5 pencils. Jerry gave her 7 more. How many pencils does Tom have altogether?	*Result Unknown* Tom had 12 pencils. She gave 7 to Jerry. How many pencils does Tom have left?	*Difference Unknown* Tom has 12 pencils. Jerry has 7 pencils. How many more pencils does Tom have than Jerry?	*Whole Unknown* Tom has 5 blue pencils and 7 yellow pencils. How many pencils does Tom have?
Change Unknown Tom has 5 pencils. How many pencils does she need in order to have 12 pencils altogether?	*Change Unknown* Tom had 12 pencils. She gave some to Jerry. Now she has 5 pencils left. How many pencils did she give to Jerry?	*Compare Quantity Unknown* Jerry has 5 pencils. Tom has 7 more than Jerry. How many pencils does Tom have?	*Part Unknown* Tom has 12 pencils. 5 are yellow and the rest are blue. How many blue pencils does Tom have?
Start Unknown Tom had some pencils. Jerry gave her 5 more. Now she has 12 pencils. How many pencils did Tom have to start with?	*Start Unknown* Tom had some pencils. She gave 5 to Jerry. Now she has 7 pencils left. How many pencils did Tom have to start with?	*Referent Unknown* Tom has 12 pencils. She has 5 more pencils than Jerry. How many pencils does Jerry have?	

Multiplication and Division

Multiplication	**Measurement Division**	**Partitive Division**
Camelia has 3 bags of cookies. Each bag contains 4 cookies. How many cookies does Camelia have all together?	Camelia has 12 cookies. She puts 4 cookies in each bag, with the same number of cookies in each bag. How many bags can she fill?	Camelia has 12 cookies. She puts the cookies into 3 bags, with the same number of cookies in each bag. How many cookies are in each bag?

Reference: *Children's Mathematics: Cognitively Guided Instruction*, Thomas P. Carpenter, Elizabeth Fennema, Megan Loef Franke, Linda Levi & Susan B. Empson, Heinemann/NCTM, 1999.

Conceptual Models for Operations

Models for addition and subtraction

The operation of addition has only one commonly identified conceptual model, that of *joining*: putting two sets together. Subtraction, however, has two: *take-away* and *matching*. We can also make further distinctions, between *static* and *dynamic* problem contexts.

1. In your small group, identify which type of word problem in the table on the last page involve(s) the matching model for subtraction, and then classify each problem type as static or dynamic.

2. Classify the following simple story problems by problem type, using the table.

1. Mehmet has 13 poker chips. Five are red and the rest are blue. How many poker chips does Mehmet have?

2. Tina has 11 tennis balls. She hits 6 of them over the fence. How many does she have left?

3. Rita has 5 dogs. How many more dogs does she need to have 13 dogs altogether?

4. Dave had 13 guitar picks. He gave 5 guitar picks to Marty. How many picks does Dave have left?

5. Yesterday, Mary had 13 teddy bears. She gave some to Denis. Now she has 5 teddy bears left. How many bears did Mary give to Denis?

6. Julie has 11 photos and Marc has 6 photos. How many more photos does Julie have?

7. Ben has 5 candies. Lauren has 8 more than Ben. How many candies does Lauren have?

8. Tim had some books. He gave 5 to Lisa. Now he has 8 books left. How many books did Tim have to start with?

9. Yesterday, Tordis had some cupcakes. Today Birgir gave her 6 more cupcakes. Now she has 11 altogether. How many cupcakes did Tordis have yesterday?

10. Jana had 5 cookies. Carl gave her 8 more cookies. How many does Jana have altogether?

11. Kelly has 5 red socks and 8 blue socks. How many socks does she have?

12. A moment ago, Val blew 11 bubbles. Some of them burst, and now she only has 6 left. How many bubbles burst?

13. Katie has 13 goldfish crackers. Sam has 5 goldfish crackers. How many more crackers does Katie have than Sam?

14. Peyton has 13 marbles. She has 5 more marbles than Maddie Cate. How many marbles does Maddie Cate have?

15. Ruth has 6 tickets left to ride rides at the amusement park. Her favorite roller coaster requires 11 tickets. How many more tickets does she need?

16. Sarah had some blocks. Adelfo gave her 5 more blocks. Now she has 13 blocks. How many blocks did Sarah have to start with?

Models for multiplication

1. For the following simple multiplication story problems, identify first (i) which conceptual model for multiplication is being invoked (set model or array model) in each problem, and then (ii) which quantities in each problem play the same role.

   (a) Dave owns 5 acoustic guitars. Each guitar has 6 strings. If Dave decides to restring all of his guitars, how many strings will he replace?

   (b) A small rectangular bedroom is 8 feet long on one side and 9 feet wide on another side. The floor is to be tiled with square ceramic tiles which are each 1 foot on a side. How many tiles will it take to complete the floor?

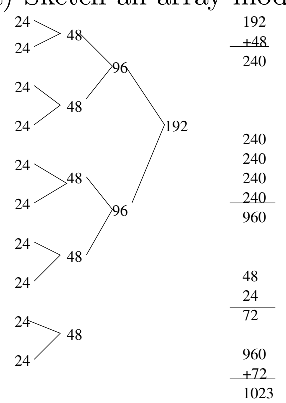
   (c) How many Life Savers would Sam eat if he ate 3 packages of Life Savers with 5 candies in each package?

   (d) There are 6 nectarines in each crate. How many nectarines are there in 5 crates?

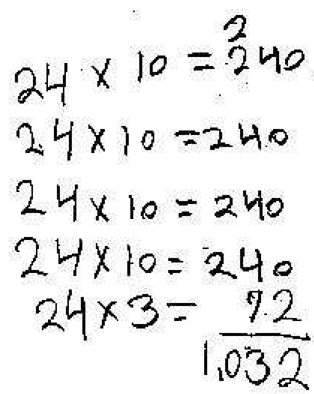
   (e) The drum corps consists of 4 rows of drummers with 8 drummers in each row. How many drummers are there in the corps?

   (f) A.J. gave a gift bag to each of his 9 friends who attended his birthday bowling party. Each bag had 7 pieces of candy. How much total candy did A.J. give out?

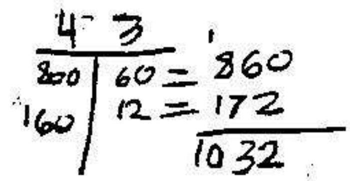
2. (a) Sketch an array model representation for each of the following student computations.



Leonora's computation for 43 groups of 24



Tim's computation for 43 groups of 24



Sean's computation for 43 groups of 24

(b) Construct a mapping between the elements of the two representations (set and array) for these computations. (What is the relationship between them?)

3. In your small group, write a paragraph addressing the following three issues:
   - Describe or explain the two conceptual models for multiplication.
   - Describe how the units of the factors and products are related in each model.
   - How are the relationships between units of factors and products different than the relationships between units of numbers in addition and subtraction models?

4. How do the following multiplication problems relate to the two conceptual models for multiplication?
   (1) A greyhound can run 18 meters per second at top speed. If a greyhound runs at top speed for 5 seconds, how far has it traveled?
   (2) Jo's quilt collection is 5 times as large as Sandy's. Sandy has 3 quilts. How many quilts does Jo have?
   (3) Jacob has 3 pairs of shorts and 4 T-shirts. How many outfits can he make with those items of clothing?

5. In story problems, why does "of" mean multiply?

Models for division

There are two conceptual models for division, derived from the set model for multiplication (there's technically a third derived from the array model, but in practice it's rarely seen). Classify the simple division problems below as to which of the two set-model factors is the divisor.

1. Adelfo has 120 toy cars and wants to divide them into 5 teams. How many cars will be on each team?

2. Sarah has 11 dresses and wears 5 each week. How many weeks will it take for her to wear them all?

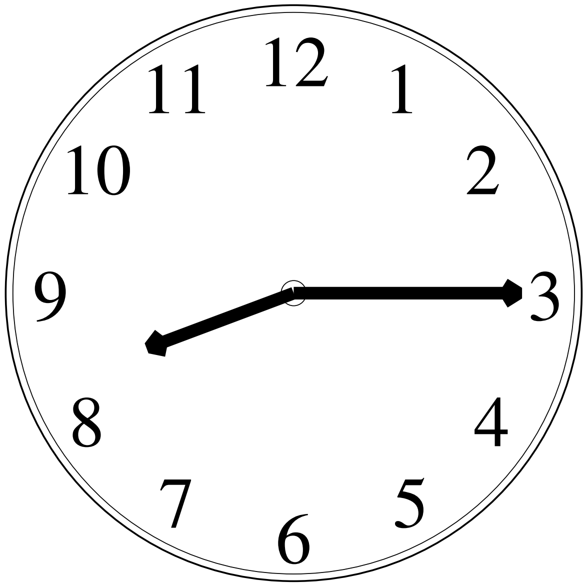
3. If Velin must eat 78 peas during the next 7 days, how many peas must he eat per day, on average?

4. A class of 72 children is going on a field trip to the zoo. If each bus holds 32 children, how many buses will be needed?

5. Dr. Jeffily, a veterinarian, needs to vaccinate 104 cows. If he can vaccinate 17 cows each day, how long will it take him to finish?

6. After carpeting a room, Shiloh has 107 square feet of carpet left over. If another bedroom is 13 feet wide, how far would her leftover carpet cover the second room (going the complete width of the room and starting from one end)?

Clock Arithmetic



In this problem we will adapt familiar arithmetic operations to work with a different set of numbers, namely those on the face of a clock. As you work, use the context of time (measured in hours—we won't worry about minutes, seconds, etc.) to guide your definitions.

1. Develop a clear definition for a new operation $\oplus$ which will be addition in "clock arithmetic", operating *only* on the numbers {1, 2, 3, ..., 12}. The first argument should be the current time (hour only) on a twelve-hour clock, the second argument should be elapsed time, and the output should be the new time (after the elapsed number of hours have passed), also on a twelve-hour clock. Make it as much like normal addition as you can (including properties). Represent your definition four different ways: using (a) words, (b) a table, (c) a diagram, and (d) a formula.

2. Explain how to calculate $3 \oplus 5$ ("three clock-plus five") and $7 \oplus 10$ using each definition.

3. If necessary, revise your definition to make it as precise as necessary, and then analyze the structure of this new "clock addition" operation—that is, describe which of the six field properties the operation and the set it operates on have (leaving out distributivity for now). Explain why it does or does not have each property.

Clock Arithmetic II

1. Define clock subtraction $\ominus$ on the set $\{1, 2, 3, ..., 12\}$, analogous to clock addition $\oplus$. Give a clear explanation of how to calculate with it, how to interpret it in real-world (time) terms, and analyze its properties.

2. Can clock arithmetic be extended to include multiplication and division operations under which the set $\{1, 2, 3, ..., 12\}$ remains closed? Why or why not?

3. Do the operations $\oplus$ and $\otimes$ form a *field*—that is, do they together satisfy all the field properties? (For the list, see "Properties of Operations".)

4. How could you interpret $\otimes$ in the context of time?

Clock Arithmetic III

1. Would the *properties* of clock addition $\oplus$ change if it were based on an 11-hour clock instead (and the numbers $\{1, 2, 3, ..., 11\}$)? (Obviously some details of the definition would change, but what about the properties?) Explain.

2. (a) Make tables to define as many of the clock arithmetic operations as possible on a two-hour clock: that is, using only the numbers 1 and 2. (b) Relate your answers to the tables you filled in for odd/even arithmetic in Session 2. How is clock arithmetic like odd/even arithmetic? How is it different?

3. How does the structure of the inverses change as the set size (number of hours on the clock) changes? How many numbers are their own inverses?

4. On clocks of what "size" (how many hours) could clock division be defined?

P.S. Mathematicians refer to clock arithmetic as "modular arithmetic", and instead of putting circles around the operation symbols, they write "mod n" after the equation, where n is the number of hours on the clock. So they would refer to "normal" clock arithmetic as "arithmetic modulo 12," and to odd/even arithmetic as "arithmetic modulo 2." Because of the answer to the last question above, modular arithmetic only forms a field when the modulus is a prime.

Definitions of even

In your third-grade class one day, a discussion arises about odd and even numbers. In order to ensure that everyone has the same understanding of these terms, you ask students to explain what an even number is. Students volunteer the following four definitions.

- "An even number means you can break it into two equal groups, with nothing left over."
- "Even means you can make it with twos, so just pairs with no leftovers."
- "It goes back and forth between odd and even: 1 is odd, 2 is even, 3 is odd, 4 is even, and keep going like that."
- "If a number ends in 0, 2, 4, 6, or 8, it's even. If it ends in 1, 3, 5, 7, or 9, it's odd."

1. Analyze each of these in terms of the underlying idea and the language in which it is communicated. Which could make (perhaps with slight revisions) productive classroom definitions for even? Why? Where does the language need adjusting?

2. How do you know that numbers that are considered even numbers by one of the first two definitions would also be considered even by the other? That is, how can you show these two definitions are equivalent?

3. Explain the mathematical relationships among the four ideas underlying the definitions given above.

4. One student says, "6 is even and odd. It's even because it's two groups of three. But 6 is also odd because it's three groups of two." Which of the four definitions of even above is he trying to use? Based on this example, what does he really mean by "even"? What kinds of numbers would satisfy his unspoken definition?

Odd and even arithmetic

1. In elementary school, children can begin to make generalizations about what happens when you operate on odd and even numbers. What is the result when you add two odd numbers? an odd number and an even number? etc. Fill out the odd/even arithmetic tables below: in each cell, place an O if the result is always odd, an E if the result is always even, and X out the whole cell if the result is not consistent.

$+$	O	E
O		
E		

$-$	O	E
O		
E		

$\times$	O	E
O		
E		

$\div$	O	E
O		
E		

2. One of the focal points of this course is determining what constitutes an argument to prove a generalization, where no number of examples is sufficient. What kinds of arguments can you make to prove your generalization (above) regarding the sum of two odd numbers? For each argument, specify (i) the definition of odd/even you are invoking, (ii) the conceptual model/definition for addition you are invoking, and (iii) the representation (see page 1) you are using.

3. What kinds of arguments can you make to prove your generalizations regarding the products of odd and even numbers? For each argument, specify (i) the definition of odd/even you are invoking, (ii) the conceptual model/definition for multiplication you are invoking, and (iii) the representation (see page 1) you are using.

4. According to your tables above, the set {odd, even} is closed under what operations?

Math 1330
UNIT THREE
Numeration Systems and Algorithms

Throughout human history cultures have developed different ways to represent numbers. To what extent does the way we perform simple arithmetic depend on the form of the number system we use? In this unit we consider some other ways to write numbers (than the Hindu-Arabic base-ten place-value system we know), and what arithmetic algorithms (procedures) look like in those other systems. We hope that you will not only gain a better understanding of arithmetic algorithms in general, but also understand the issues your students will confront when trying to add, subtract, multiply and divide large numbers for the first time.

Numeration Systems

Throughout history each culture has developed its own way to keep track of numbers. Each of these numeration systems has some distinct features, and some features in common with other systems. First, each system uses special words or symbols to represent numbers. However, in deciding what words or symbols you want to use, you also have to decide whether location or order of the words/symbols affects meaning. Most numeration systems can be classified as one of the following two types of system:

Symbol value: each symbol represents the same number, regardless of where it appears.
example: Roman numerals — the Vs in XVIII and MCMLV always mean five

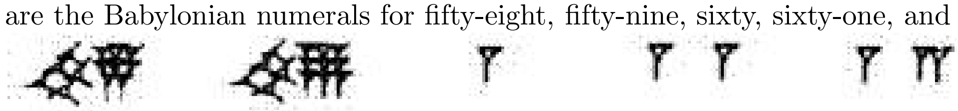
Place value: each symbol represents groups of a size that depends on where it appears.
example: Hindu-Arabic numerals — the 5s in 135 and 5042 mean different amounts

Historical note: The original Roman system used IIII for four and VIIII for nine. The inversion that led to IV and IX was a later innovation which we shall not use here.

1. Convert the following base ten numerals into Roman numerals, and then find their sum *without* converting back to base ten: 64, 137, 300.

2. The Babylonians used a sexagesimal (base-sixty) positional numeral system inherited from the Sumerians (whom they replaced about 2000 BC). Babylonian numerals were written in cuneiform, using a wedge-tipped reed stylus to make a mark on a soft clay tablet which would be exposed in the sun to harden to create a permanent record (or recycled before hardening). The Sumerian-Babylonian system used only two symbols: a vertical (downward) wedge  to represent the number one, and a horizontal (leftward) wedge  to represent ten. Sexagesimals still survive to this day, in the form of degrees, minutes, and seconds in the measurement of time and angles.

   Here are the Babylonian numerals for fifty-eight, fifty-nine, sixty, sixty-one, and sixty-two:

   Explain how to interpret each of these numerals. Note how the ones and tens are stacked in the first two numerals. How is place value invoked in the latter three numerals?

3. Convert the following base ten numerals into Babylonian numerals, and then find their sum *without* converting back to base ten: 64, 137, 300.

4. List the advantages and disadvantages of representing and adding numbers in each of the two types of system.

5. The ancient Babylonian tablet below gives the multiplication tables for what number?

   (The symbols in the first two columns [except for the first row] mean "the number multiplied by".)

Zeroes and Ones

1. Develop a *symbol value* numeration system (way to represent all whole numbers) using only two symbols: the digits 0 and 1. Describe how it works, and illustrate it by listing how to write the first twelve whole numbers in this system. (You may not use spaces, commas, accents, or any other characters or marks to give meaning to your numerals.)

2. Now develop a *place value* numeration system using only the digits 0 and 1. Describe how it works, and illustrate it by listing how to write the first twelve whole numbers in this system. (Again, do not use spaces, commas, or any other characters to give meaning to your numerals.)

3. What is the 32nd number in each list?

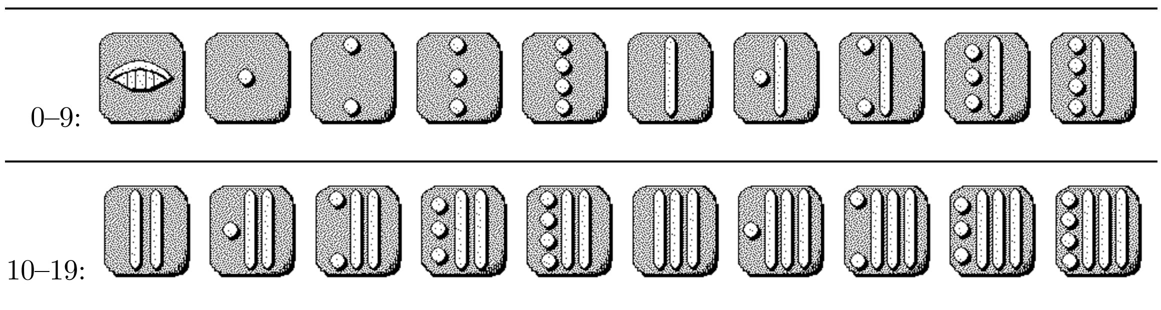
4. What is the 261st number in each list?

5. Show how to add two numbers using each of your schemes. Can you do it without converting to base ten and back?

6. What advantages and weaknesses does each coding scheme have?

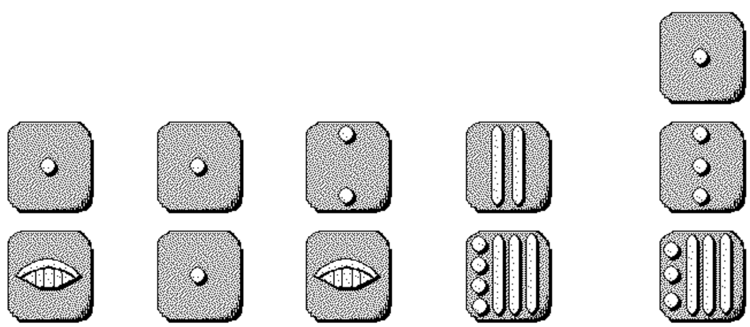
Sticks and Stones

The ancient Mayans of Central America used a number system which was a *vertically-oriented* (i.e., top to bottom instead of left to right), base-twenty place-value system. However, rather than using twenty different symbols to write the digits from zero to nineteen, they had only three symbols, which they used somewhat like the Romans in denoting values within each place. The Mayans were one of only three ancient cultures which developed the notion of zero (important in place value systems!), represented here by a shell. Each stone represents *one*, while each stick represents *five*. Below are the Mayan representations for the numbers zero through nineteen. (We line the sticks and stones up in this way to make them easier to read, but the Mayans just put them together in little piles when actually doing math.)



1. To write numbers twenty or higher, they used multiple digits, just as we do in base ten for numbers ten or higher. If the Mayan base was twenty instead of ten, what would their place values be? (instead of ones, tens, hundreds, thousands, etc.)

2. Here are some Mayan multi-digit numbers. Explain why these are correct representations in the Mayan system for the quantities given. (Remember that the Mayan system is vertical rather than horizontal, so the ones place is on the bottom, with the place values getting higher as we move up.)



twenty — twenty-one — forty — two hundred nineteen — four hundred seventy-eight

3. Convert the following numbers from Hindu-Arabic to Mayan:

   (a) 32, (b) 45, (c) 100, (d) 399, (e) 1000, (f) 1,000,000

Introduction to Bases

As you may know, our normal way of representing numbers is called "base ten", because it uses ten digits. We count from 0 to 9, and then we write 10, representing 1 group of ten and 0 leftover units. The next number, 11, represents 1 group of ten and 1 unit. This can continue indefinitely; the number 43,507 represents 4 groups of ten thousand, 3 groups of one thousand, 5 groups of one hundred, 0 groups of ten, and 7 units.

If we use a different base, the group size changes, but the general organizational principles remain the same. For example, if we take eleven and represent it in a *base three* system, then we no longer write "11" (one one), because there is no longer a tens place. Instead, the place values in a base three system are powers of three, so to the left of the ones place, there is a *threes* place, and to the left of that a *nines* place (a nine is three groups of three, just as a hundred is ten groups of ten). We can make three groups of three in eleven, with two ones left over, and we might be tempted to then write eleven as "32" in base three, meaning 3 groups of three, plus 2 ones, but *this is illegal* — we can no more put "three" in the threes place in base three, than we can put "ten" in the tens place (or any single place) in our base ten system. Instead, we take those 3 threes and make a nine out of them, so that eleven is packaged as follows: 1 group of nine, no groups of three, and 2 ones left over. We write this as "102" in base three.

In order not to confuse ourselves or others, we do two things to make it clear that we are talking about a base-three numeral and not the quantity "one hundred two": first, we write the numeral with a subscript to indicate the base, like this — 102_{three} — and second, when reading it aloud, we say, "One zero two, base three," and never use base-ten words like "hundred" or "thousand" when reading a numeral in a base other than ten.

Here are some basic rules for place value systems which hold true for any base:

- The allowable sizes of groups are powers of the base: first ones, then groups of size the base, then the base squared, the base cubed, etc. These correspond to the place values.

- The maximum number of groups of any size which you can have is *the base minus one*. If at any point you have as many groups of something as the base, then you must take them and group them together into one of the next higher sized group.

- The allowable digits to use are therefore 0 (zero) through one less than the base. (We will have no digits to represent any quantity larger than that because we will never have more than that many groups of any size.)

- To represent (group) a number of things according to a given base, begin by figuring out the largest possible group you can make using (some of) the items you have. Make as many of that size group as possible; then make as many of the next smaller size group as possible with the leftovers, and continue down the group sizes until you have just leftover ones.

(Bored yet? Hang in there, we're ready to start now. Turn the page.)

1. Perhaps the best way to get a feel for other bases is to practice grouping. First, make a set of place value mats for your group: Take some sheets of paper (not necessarily full-size) and label them as follows: ones (base^0), stacks (base^1), squares (base^2), cubes (base^3), stacks of cubes (base^4), squares of cubes (base^5), cubes of cubes (base^6). Now arrange them from right to left in the order given above.

   Take thirty-seven counters. Those can be grouped into 3 groups of ten (on the "stacks" mat) and 7 units (on the "ones" mat). That's why this number is written 37 (base ten). But they could also be grouped into 4 groups of eight (on the "stacks" mat) and 5 units (on the "ones" mat). So we could write this number as 45 (base eight), or 45_{eight}. By grouping your counters, write this number in the bases two through twelve below.

   Can you estimate in which base you will first have to write a three-digit number to represent thirty-seven?

Base twelve:	Base eight: 45_{eight}	Base four:
Base eleven:	Base seven:	Base three:
Base ten: 37_{ten}	Base six:	Base two:
Base nine:	Base five:	Base thirty-seven:

2. Now reflect on how changing the base of your number system affected the properties of your system.

   (a) How many symbols are allowable in a base n place value system? Why? What are they?

   (b) What are the values of the places in a base n place value system?

3. What is the role of zero in a place value system?

4. Write a general set of instructions for representing multidigit numbers *concretely* in a base n place value system.

5. Write a general set of instructions for representing multidigit numbers *symbolically* in a base n place value system.

6. Compare the two sets of instructions you wrote above, and identify which steps correspond to each other.

Regrouping in Other Bases

1. Now that you are good at grouping, try adding numbers concretely. Take thirty-seven and fifteen, and put both in base ten groupings (we'll start by adding in base ten). Now, start by adding the units. There are 7 units and 5 units, for a total of twelve. This is too many for our base, so we take a group of ten and move them over. Now we add the tens. There are $3 + 1 + 1$ groups of tens, for a total of 5. This number is less than our base, so we leave it. We now have 5 groups of ten and 2 units, so $37 + 15 = 52$ in base ten.

Once you understand this idea, do the same (adding thirty-seven and fifteen) in the following bases, **using your manipulatives**:

(a) base eight (b) base four (c) base two

2. There are two common concrete subtraction models: the take-away model (which is what probably comes to your mind when you think of modeling subtraction), where the quantity being subtracted is removed from the original number, and the matching model, where the two quantities are aligned beside each other, and the difference is given by the quantity in the larger number that does not have anything corresponding to it. Use both models to solve the subtraction problem thirty-seven *minus* fifteen in the same bases as above, **using your manipulatives**:

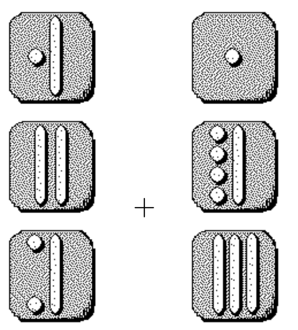
(a) base eight (b) base four (c) base two

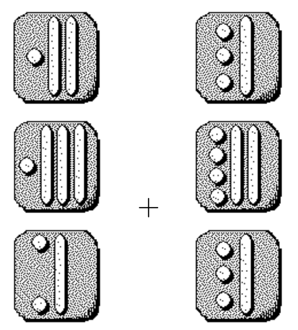
3. Compare the *processes* (obviously the results should be the same) and insights gained from using the take-away model vs. the matching model for subtraction.

4. How is regrouping in other bases like regrouping in base ten? How is it different?

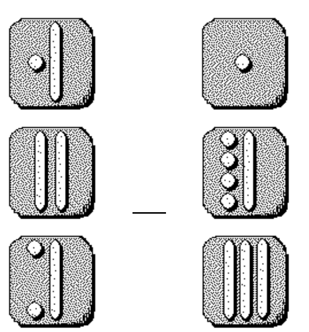
Mayan addition and subtraction

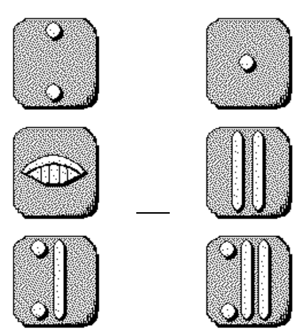
1. Addition in the Mayan system was similar to the traditional addition algorithm in our Hindu-Arabic system: first, place the two numbers next to each other, with the place values aligned; then, to add, just put the quantities in each place together — in this case, literally, by moving the sticks and stones in each place together into a single pile — and regroup when necessary. Perform the following addition problems *using models* (don't jump straight to pencil and paper), keeping in mind that regrouping ("carrying") means trading twenty in one place for one in the next highest place. Do not convert to Hindu-Arabic, except to check your answers.

(a) 

(b) 

2. Subtraction in the Mayan system, as in the Hindu-Arabic system, is performed in a way similar to addition. Try the following subtraction problems, again using models, and keeping the twenty-to-one regrouping in mind.

(a) 

(b) 

3. Recalling the kinds of strategy sequences children develop for addition and subtraction in our base-ten system, what kinds of regrouping were Mayans likely to develop when using their system to compute?

Driving in Septobasiland

The king of Septobasiland has proclaimed that the digits 7, 8 and 9 are never to be used in Septobasiland upon pain of death. As a result, the little wheels on the odometers of all the cars in Septobasiland have just the digits 0, 1, 2, 3, 4, 5, 6 on them. Thus, if the odometer registers 0006 and you drive one more mile, the odometer will register 0010, and if the odometer registers 0016 and you drive one more mile, it will register 0020.

1. If the odometer registers 0066 and you drive one more mile, what will the odometer register?

2. If the odometer registers 0325, how many miles has the car gone?

3. How many miles will the car have gone when the odometer turns over to all zeroes again?

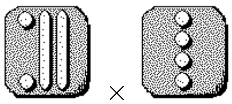
4. After a car has gone nine hundred miles, what will its odometer register?

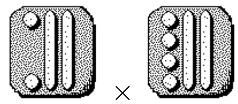
5. You and your companions go on a trip in Septobasiland, travelling together in two cars. The first car, which is new, starts with its odometer reading 0000, and the second starts with the odometer reading 1435. At the end of the trip, the odometer of the first car registers 0324. What will the odometer of the second car register?

6. Suppose again that the odometers of the two cars register 0000 and 1435 at the beginning of the trip. But this time at the end of the trip the odometer of the second car registers 2153. What will the odometer of the first car register?

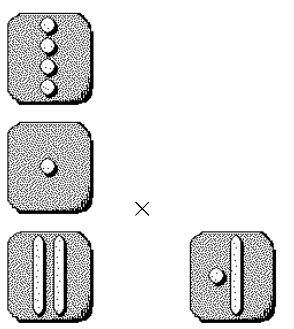
Mayan multiplication

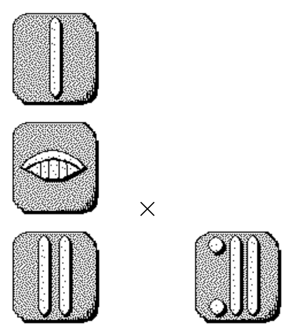
1. How would you define "multiplication"? Use this simplest definition to develop *concrete models* for these single-digit Mayan multiplication problems.

   (a) 

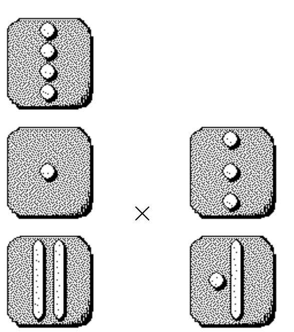
   (b) 

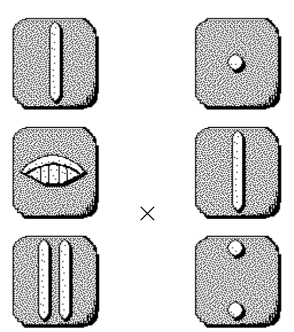
2. What would happen if you tried the same modeling approach on the multidigit multiplication problems below? How can you adapt the traditional multidigit multiplication algorithm (known as "partial products") for the Hindu-Arabic system to solve these problems? Try it out.

   (a) 

   (b) 

3. How do you have to adapt the partial products algorithm when the second factor becomes multidigit as well? Try it on these problems.

   (a) 

   (b) 

Making Groups in Other Bases

Develop and sketch *concrete* models for the following multiplication problems.

(a) a proportional model for $102_{three} \times 12_{three}$

(b) a nonproportional model for $102_{three} \times 12_{three}$

(c) a proportional model for $223_{five} \times 4_{five}$

(d) a nonproportional model for $6A_{twelve} \times 12_{twelve}$

Division in Other Bases

Most division models involve equal sharing of a total among several groups. However, the actual division process varies greatly at the concrete level, depending on whether the divisor represents the *number of groups* or the *size of each group*. Write "twenty divided by four" in bases three and four, and draw a picture of what the completed division would look like in each base, for each of the two types of division problem.

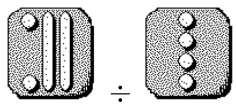
	base three	base four
"twenty divided by four is five" in numerals:		
concrete [proportional] representation for twenty:		

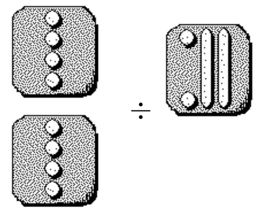
Measurement division
(twenty divided
into groups of four;
how many groups?)

Partitive division
(twenty divided
into four groups;
how many in each group?)

Mayan division

1. How would you define "a divided by b"? Use your simplest definition to develop *concrete models* for these Mayan division problems.

   (a) 

   (b) 

2. The same two division problems can be represented as follows in ternary (base three). Develop models for these representations as well.

   (a) $110_{three} \div 11_{three}$

   (b) $10010_{three} \div 110_{three}$

3. How high does someone working in a large base (such as the Mayans' twenty) need to know multiplication tables in that base? What about someone working in a small base? What do you do when you don't know the multiplication tables for the divisor?

Money in Septobasiland

1. Of course, the digits 7, 8 and 9 are also barred from the currency, which changes the way people count money. For example, there are only forty-nine Septocents in a Septodollar. (Why?) Now suppose you walk into a bank in Septobasiland with a bill that says "$100" and ask the cashier for $1 bills. How many will the cashier give you? Now suppose you trade all of those $1 bills in for Septocents. How many Septocents will you receive?

2. If the minimum hourly wage in Septobasiland is "$6.34", how much does a person on minimum wage earn before taxes in a five-day work week (working eight hours per day)?

3. The sign at a gas station says that gasoline costs "43" Septocents per liter. How much would "24" (eighteen) liters of gas cost? (Work directly in the Septobasiland number system.)

4. Suppose you want to rent a car to go on a trip. An advertisement says that in Septobasiland you can rent a car for five days for $303. How much is this per day?

5. The sign on the highway reads "Septobasiland City 2626 km". According to the owner's manual, your car gets 13 km to the liter on the highway, and the tank can hold 100 liters. How many tankfuls will the trip require? (Note all numbers are in Septobasiland notation.)

Arithmetic in Other Bases

If you haven't done so already, discuss whether (and how) the four basic arithmetic algorithms with which you're familiar extend to other bases. Then try to work out the problems below, *without* converting to base ten and back. Work directly in the given bases, and think about what you're doing.

1. (a) base six

$$\begin{array}{cccc} 4 & 0 & 4 & 3 \\ + & 3 & 1 & 3 \\ \hline \end{array}$$

(b) binary (base two)

$$\begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 1 \\ + & 1 & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

(c) hexadecimal (base sixteen)

$$\begin{array}{cccc} & 3 & A & E \\ + & B & 0 & 5 \\ \hline \end{array}$$

2. (a) base eight

$$\begin{array}{cccc} 2 & 6 & 1 & 3 \\ - & 7 & 0 & 4 \\ \hline \end{array}$$

(b) binary (base two)

$$\begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

(c) base eleven

$$\begin{array}{cccc} 2 & 3 & A & 5 \\ - & 6 & 0 & 6 \\ \hline \end{array}$$

3. (a) ternary (base three)

$$\begin{array}{cccc} 2 & 0 & 2 & 1 \\ \times & & 1 & 2 \\ \hline \end{array}$$

(b) base nine

$$\begin{array}{ccc} & 8 & 8 \\ \times & 4 & 6 \\ \hline \end{array}$$

4. (a) heximal (base six)

$$13 \overline{)\,5\;4\;3\;0}$$

(b) binary (base two)

$$110 \overline{)\,1\;0\;1\;1\;0\;1}$$

Note: In bases larger than ten, the characters A, B, C, D, E, F represent the numbers ten through fifteen.

Math 1330
UNIT FOUR
Fractions

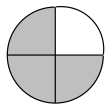
In this unit we look at fractions, one of the most important topics in the math of the upper elementary grades, and one whose underlying concepts come in much earlier. We will take an especially close look at the correspondence between fractions and decimals.

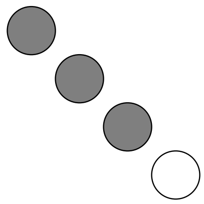
Representations of Fractions

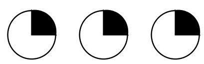
1. In your small group, develop usable definitions for "fraction" and "decimal". Try to strike a balance between completeness (include multiple different meanings) and simplicity (would K–8 students understand the definitions?). Write the definitions down (here or in your notebook).
2. In your small group, brainstorm a list of context *types* in which fractions are used, and classify them by the dimensions of the units involved, or by the properties of the resulting model.
3. Fill in the table below to analyze and distinguish the properties of the various classroom manipulatives commonly used to represent fractions.

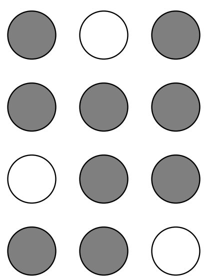
Manipulative	**Context**	**Denominators**	**Other Properties**
pattern blocks			
Cuisenaire rods			
fraction circles & squares			
paper folding			
paper plates			
chips/counters			
egg cartons			
other (specify)			

4. Which of the following could represent 3/4? Explain in each case.
(adapted from problems used by Deborah Ball at the University of Michigan)

(a) 

(b) 

(c) 

(d) 

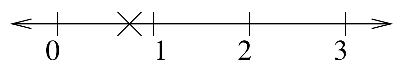
(e) How many 4–oz. chocolate bars can be made with 3 oz. of chocolate?

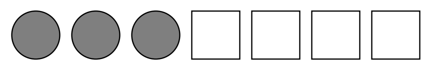
(f) 18 crayons of a box of 24

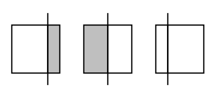
(g) 21 boys and 28 girls in kindergarten

(h) 0.75

(i) sharing 3 bottles of soda equally among 4 people

(j) 

(k) 

(l) 

5. In your small group, identify the different kinds of symbolic representations that are used for rational numbers, and the contexts in which each is typically used.

6. A *complex fraction* is a number written in fractional form a/b where a and/or b are not whole numbers (but b is still not zero). One example is the square root of 1/2, usually written either $1/\sqrt{2}$ or $\sqrt{2}/2$ (why are these equivalent? why rationalize the denominator?). Another is in the argument given by a student in one of this semester's readings, in which he argues that 1/8 can be viewed as $1\frac{1}{8}/9$.

   Are complex fractions rational numbers? Are they at all meaningful? What are some other examples of complex fractions?

7. Would representations of fraction (at any level) be different in other bases?

The Bagel Problem

Given the problem

> If three dozen bagels are shared equally among five people, how many *dozens of bagels*, or how much of a dozen, will each person get? What fraction of the whole is this?

1. What is the solution to the problem? How do you know?

2. How would the problem be different if it had been three feet of ribbon being cut into five equal strips? How would it have been different if it had been three dozen eggs being split into five equal groups? What other units (than dozens of bagels) might change the way one approaches the problem — or the answer(s) one can give?

3. More generally, what is the role of context and unit in the bagel problem?

4. What other mathematical ideas or issues are raised in this problem — that is, what are the pieces or elements that affect the solution?

5. Try to rescale the problem so that it involves essentially the same mathematical issues you identified above, but is somehow simpler or easier.

6. Try to rescale the problem so that it involves the same ideas, but is somehow more complex or harder. Why is it harder?

7. Can you make a map between different representations (or solutions) to the original problem? (This will need to be done in a group setting.)

8. Write a generalized statement which describes the result of this kind of sharing problem, in terms of the quantities involved.

The Cookie Jar Problem

1. There was a jar of cookies on the table. Amanda, who was hungry, ate half the cookies. Then Beth came along and ate a third of what was left in the jar. Next Christine came by and decided to take a fourth of the remaining cookies with her to class. Then Danielle dashed up and took a cookie to eat. When Eleanor looked at the cookie jar, she saw that there were two cookies left. How many cookies were there in the jar to begin with? Explain.

2. The two problems below are rescaled versions of the problem above. How do the different numbers change the nature of the problem? Or do they?

   (a) There was a jar of cookies on the table. Amanda, who was hungry, ate one seventh of the cookies. Then Beth came along and ate a third of what was left in the jar. Next Christine came by and decided to take a fifth of the remaining cookies with her to class. Then Danielle dashed up and took four cookies to eat. When Eleanor looked at the cookie jar, she saw that there were four cookies left. How many cookies were there in the jar to begin with?

   (b) There was a jar of cookies on the table. Amanda, who was hungry, ate one eighth of the cookies. Then Beth came along and ate a seventh of what was left in the jar. Next Christine came by and decided to take a third of the remaining cookies with her to class. Then Danielle dashed up and took five cookies to eat. When Eleanor looked at the cookie jar, she saw that there were five cookies left. How many cookies were there in the jar to begin with?

Law and Order

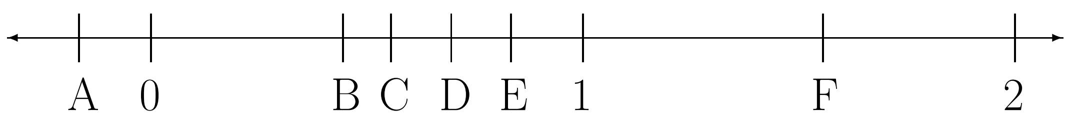
1. Suppose that $a > 1$, $0 < b < 1$, and $0 < c < 2$. Fill in each box with $<$, $=$, $>$, or NMI (Need More Information).

(a) $a \cdot b \; \square \; a$ (b) $b \cdot c \; \square \; a$ (c) $a \cdot b \cdot c \; \square \; b$

(d) $a + b \; \square \; a$ (e) $a + c \; \square \; a$ (f) $a + c \; \square \; b$

(g) $\frac{c}{c} \; \square \; c$ (h) $b^2 \; \square \; b$

2. Pictured below is a number line with some identified points on it. Use this number line to answer the questions in (a)–(e).



(a) If the numbers represented by the points D and E are multiplied, what point on the number line best represents this product?

(b) If the numbers represented by the points C and D are divided, what point on the number line best represents the quotient?

(c) If the numbers represented by the points B and F are multiplied, what point on the number line best represents the product?

(d) Suppose 20 is multiplied by the number represented by E on the number line. Estimate the product.

(e) Suppose 20 is divided by the number represented by E on the number line. Estimate the quotient.

Find the Fraction

Find a fraction whose numerator and denominator are two-digit numbers so that it is as close as possible to but not equal to $\frac{9}{10}$. You may not use any digit twice. Example: $\frac{19}{21}$ is close to $\frac{9}{10}$, but it is not allowed because it uses the digit 1 twice.

The Condominium Problem

In an adult condominium complex, $\frac{2}{3}$ of the men are married to $\frac{3}{5}$ of the women. What portion of the residents are married? (Assume men are married only to women, and vice versa, and that married residents' spouses are also residents.)

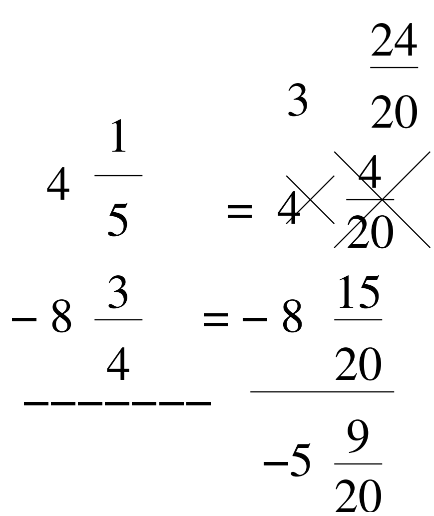
Adding and subtracting fractions

1. Suppose that you don't know the traditional algorithms for operating on fractions. How can you find a number to represent the sum $2/3 + 1/2$? How about $2/3 - 1/2$?

2. Give the most elementary solutions (not just answers) you can for the following sums. What are the important distinguishing features of each?

   (a) $\frac{1}{7} + \frac{3}{7}$ (b) $\frac{1}{2} + \frac{1}{4}$ (c) $\frac{1}{2} + \frac{1}{3}$ (d) $\frac{1}{2} + \frac{2}{3}$ (e) $\frac{3}{4} + \frac{1}{10}$

3. Analyze the following student work in detail.



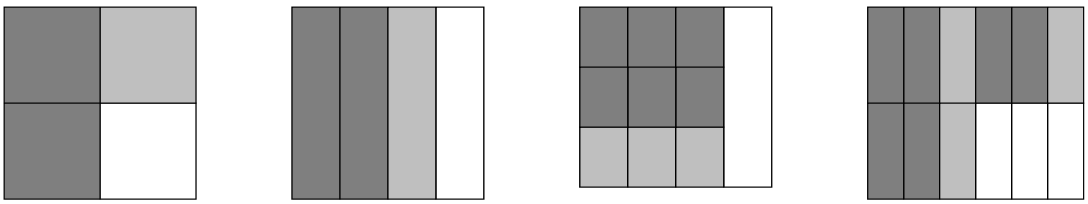
Multiplying fractions

1. (a) Is "1/2 of 2/3 miles" the same as "2/3 of 1/2 mile"? Explain.

   (b) How (other than the traditional algorithm) would you find 1/2 of 2/3 miles? How would you find 2/3 of 1/2 mile?

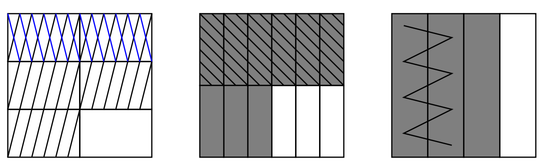
   (c) What is the relationship between your answers to (b) and the traditional algorithm for multiplication of fractions?

2. Which of the following give an *array* model representation for "2/3 of 3/4"? a set model?

   How do they differ from the array model of multiplication as originally developed for whole numbers?

3. A paper presented by Andrew Izsak at PMENA 2006 examines the classroom practice of two sixth-grade teachers. During class discussion, one of them encountered the three student representations given below. She pronounced the first and third correct, and the second incorrect. Which of her analyses were correct, and which incorrect?

   (a) $\frac{1}{3}$ of $\frac{5}{6}$ (b) $\frac{2}{3}$ of $\frac{3}{4}$ (c) $\frac{2}{3}$ of $\frac{3}{4}$

4. What complicating factor is common to the first two student sketches above? What complicating factor is particular to the multiplication (not the student representation) in (a)?

Oranges and Posters

Develop and sketch concrete models to solve the following problems.

1. You have $2\frac{1}{2}$ oranges. If each student serving consists of 3/4 oranges, how many student servings (or parts thereof) do you have?

2. You have $1\frac{1}{2}$ oranges. If this is enough to make 3/5 of an adult servings, how many oranges constitute 1 adult serving?

3. Sarah is making posters by hand to advertise the school play, but the posters she has designed are not the same size as a standard sheet of paper. She has $3\frac{1}{2}$ sheets of paper left, which is enough to make $2\frac{1}{3}$ posters. How many sheets of paper does each poster use?

4. If Alberto is also making posters, but his posters only use 2/3 of a sheet of paper, how many of Alberto's posters will those 3 1/2 sheets of paper make?

Dividing Fractions

1. (a) Use the traditional long division algorithm to calculate $53 \div 4$.

   (b) Now use the traditional long division algorithm to calculate $12\frac{1}{2} \div \frac{1}{3}$ (without converting everything to decimals).

   (c) What is the relationship between these two computations?

2. Write a short "story problem" that models the division $2\frac{1}{2} \div \frac{2}{3}$. Is the answer to your question really $2\frac{1}{2} \div \frac{2}{3}$?

3. Suppose you're baking cookies and have lots of all of the ingredients except for sugar, of which you have only 6 cups. Each batch of cookies requires 3/4 cup of sugar. How many batches can you make? Try to solve this problem in several different ways.

The Sandwich Problem

Sue's mom has enough ingredients to make 5 sub sandwiches for a party that Sue is hosting. Sue has not invited any friends yet because she wants to make sure that there are enough snacks for her and each of the friends that she invites. The decision that Sue and her mom need to make involves what fraction of each sub sandwich should constitute a "serving" for Sue and each of the party guests.

1. First, they wonder how many servings there will be if 1/2 of a sandwich is a serving.

   (a) Write the division computation that answers their question; attach either "sandwich" or "serving" as a unit label to each of the three numbers in your computation, and explain why this computation (and division in general) is appropriate for this context.

   (b) Write a multiplication computation that answers their question and briefly explain why this computation (and multiplication in general) is appropriate for this context. ("Because you invert and multiply" is not a valid explanation!)

2. Next, they wonder how many servings there will be if 1/3 of a sandwich constitutes a serving. Repeat parts (a) and (b) of question 1 for 1/3 of a sandwich as a serving.

3. Next, they wonder how many servings there will be if 2/3 of a sandwich constitutes a serving. Repeat parts (a) and (b) of question 1 for 2/3 of a sandwich as a serving.

4. Next, they wonder how many servings there will be if 3/5 of a sandwich constitutes a serving. Repeat parts (a) and (b) of question 1 for 3/5 of a sandwich as a serving.

5. Generalize your conclusions by repeating parts (a) and (b) of question 1 one last time, now using the expression m/n to represent that fraction of the sandwich that constitutes a serving.

Dividing Fractions Problem Bank

As a teacher, you will need to write and adapt your own division of fractions problems for your students, as you identify their particular needs. I have found that it is particularly difficult to write good partitive division of fractions problems (as compared to measurement), perhaps because the idea of the number of groups not being a whole number is a tricky one to frame in a realistic context. Also, keep in mind some fruitful real-world contexts in which fractional quantities naturally arise.

1. Rajiv has a cookie recipe that calls for 2/3 cup of sugar per batch. If he has $1\frac{1}{2}$ cups of sugar, how many batches of cookies can he bake?
2. $3\frac{1}{2}$ cups of lemonade will fill 2 1/3 glasses. How many cups of lemonade does each glass hold?
3. Carmen is tying ribbons in bows on flowers. She uses $2\frac{1}{4}$ inches of ribbon on each flower. If she has $7\frac{1}{2}$ inches of ribbon left, how many bows (or parts of a bow) can she make?
4. Pat is also tying ribbons into bows. Pat sees the same $7\frac{1}{2}$ inches of ribbon measured out and says, "Since my bows are bigger than Carmen's, that's only enough for me to make $2\frac{1}{4}$ bows." How much ribbon does Pat use on each bow?
5. How many days (and parts of a day) will it take Kate to write a $17\frac{1}{2}$-page story if she writes $3\frac{3}{4}$ pages a day?
6. How many pages per day must Kate write on average if she is to write a $10\frac{1}{2}$-page story in $3\frac{1}{2}$ days?
7. If Kit can mow one lawn in 1 2/3 hours, how many lawns (including parts of a lawn) can she mow in $7\frac{1}{2}$ hours?
8. Kit swam one lap in the pool in $3\frac{1}{2}$ minutes. During that same time Jennifer swam $1\frac{1}{2}$ laps. How many minutes did Jennifer take (on average) to swim one lap?
9. Dane has enough paint left to cover 40 square feet of wall. If the wall he is painting is $3\frac{1}{2}$ feet high, how far along the wall can he paint?
10. A developer wants to subdivide a plot of land which covers $4\frac{1}{2}$ acres into smaller plots which cover 2/5 acre each. How many of the smaller plots can he make?

Classify each of the above problems by type. What important distinguishing characteristics do you see in them?

Fractions to Decimals

1. Write down any non-integer rational number both as a common fraction (i.e., in a/b form) and as a decimal. How can you prove that these two representations of the rational number are the same?

2. (a) Decimal representations of numbers in general may terminate (like 0.5), may repeat forever (like $1/3 = 0.\bar{3} = 0.333333...$), or may do neither (i.e., go on forever without repeating, like the digits of $\pi = 3.1415926535...$). Convert the fractions below into decimals, and classify each fraction as terminating, repeating, or neither. Be ready to convince others how you know for sure.

(i) 1/4
(ii) 1/5
(iii) 1/6
(iv) 1/7
(v) 1/8
(vi) 1/9
(vii) 1/10
(viii) 1/11
(ix) 1/15
(x) 1/16

(b) In general, given a fraction in simplest terms a/b, how can you tell (without doing the actual division) if the equivalent decimal representation will terminate?

3. (a) Can 1/7 be written as a repeating decimal? How can you be sure?

(b) Can 5/17 be written as a repeating decimal?

(c) What is the key to showing that a given fraction can be written as a repeating decimal? What does it say about the maximum length of the repeating digit string in that case?

(d) Based on your answers above, what can you say about nonrepeating, nonterminating decimals? What can you say about decimal representations of fractions?

Decimals to Fractions

1. Try to write each of the following decimals as a common fraction.

   (a) 0.432

   (b) $0.\overline{1} = 0.11111...$

   (c) $0.0\overline{1} = 0.011111...$

   (d) $0.\overline{01} = 0.010101...$

   (e) $0.\overline{123} = 0.123123123...$

   (f) $2.2\overline{43} = 2.2434343...$

   (g) $0.939339333933339...$

2. Can all decimals (terminating, repeating, and the rest) be written as common fractions? Explain how you know, and describe a general procedure for converting, in the cases when it's possible.

Going to Extremes

1. Given your knowledge about the correspondence between common fractions and repeating decimals, can you write $0.\overline{9} = 0.99999...$ as a common fraction?

2. Is there a largest...

   (a) integer?

   (b) real number?

   (c) number less than 1?

3. Is there a smallest positive number (greater than zero)?

Place value in other bases

1. Place values are defined as powers of the base: the numeral 110 represents one hundred and (one) ten in base ten, but represents one four and one two in base two, one nine and one three in base three, etc. Fill out the table below using words (not numerals) to describe the value of each place value in the following bases.

Base	1000	100	10	1	0.1	0.01	0.001
ten	thousands	hundreds	tens	ones	tenths	hundredths	thousandths
two							
three							
four							
five							
twelve							
twenty							

What pattern do you notice?

2. Represent the quantity three-fourths as a "decimal"[1] in the following bases:

(a) base two (b) base four (c) base twelve (d) base twenty

[1]An unfortunate term here, as the word *deci*mal refers inherently to base ten.

3. Develop concrete (or semi-concrete) models for 3/4 which illustrate the equivalence of the representations developed in the previous question. (In considering how to show equivalence of quantities cut into different sizes, you may want to recall how we compare fractional representations of rational numbers.)

4. Represent the quantity $3A.39_{twelve}$ in bases two, four, ten and twenty.

5. Look back on the procedures you wrote down for representing concrete and symbolic place value representations for whole numbers at the end of "Introduction to Bases" in the previous course. How must these procedures be adapted or extended for rational numbers?

Fractions and decimals in other numeration systems

1. Represent the fraction 2/3 as a "decimal" in (a) base three, (b) base nine, (c) base two, (d) base twenty.

2. In which bases can the fraction one half be written as a terminating (not repeating) "decimal"? One fourth? One twelfth?

3. In general, given a fraction a/b, in which bases will the fraction's "decimal" representation terminate? Why?

Models for multiplying decimals

1. Build and sketch (a) a set model, and (b) an array model, for the multiplication 1.2×3.5. Underneath each sketch, write a sentence explaining what "1.2×3.5" means in the corresponding context.

2. What moves students away from the desire to use concrete models for multiplying decimals?

3. Now apply the traditional symbolic algorithm to the multiplication 1.2×3.5. How does each of the parts shown in your work correspond to parts of the two models you built above?

4. (a) How is the traditional algorithm for multiplying decimals different than the traditional algorithm for multiplying multidigit whole numbers? (b) Why is this mathematically valid? (c) What is the underlying motivation for doing this?

Math 1330
UNIT FIVE
Number Theory

Number theory involves the structure of numbers, including factors and prime numbers, which are the parts on which we will concentrate here. The hope here is that by working through the following problems and discussions you will see some order and organization in the way numbers are made, and gain some intuition about the underlying structure.

Factors & Prime Numbers

You may have heard the terms *prime* and *composite* before. These words talk about how numbers can be broken down in terms of *factors*. Factors are numbers (usually whole numbers) which divide evenly into a given number — or, to put it another way, numbers which are multiplicative building blocks of larger numbers. For example, the factors of 6 are 1, 2, 3 and 6, because we can write $1 \times 6 = 6$ and $2 \times 3 = 6$. The factors of 49 are 1, 7 and 49 (we don't list 7 twice). A number is *prime* if its only factors are 1 and itself. The first prime number is 2 (we don't count 1). A number is *composite* if it is not prime (again, we usually do not classify 1 in either category). Note that although 1 is not classified as prime or composite, it *does* count as a factor of every number.

The *prime factorization* of a number involves writing the number as a product of primes, e.g., the prime factorization of 60 is $2 \times 2 \times 3 \times 5$ or $2^2 \times 3 \times 5$.

0. At home, factor the numbers from 1 to 100. Make a table with four columns: the number, a list of all its factors, the number of factors it has, and its prime factorization. For example, the line for twelve should look like this:

12	1, 2, 3, 4, 6, 12	6	$2^2 \cdot 3$

   Use the table to help you answer the following questions.

1. What is the first composite number?

2. Are there any other even primes besides 2?

3. List the first ten prime numbers.

4. If p is a prime number, what are the factors of p^4, in terms of p?

5. Can you find a number smaller than 100 with exactly one factor? Exactly two factors? three factors? What is the first number n for which no number from 1 to 100 has exactly n factors?

6. What number less than 1,000 has the most factors?

GCF & LCM

The *greatest common factor*, or GCF, of two numbers, is the largest factor they share in common. GCF(12,18)=6. The *least common multiple*, or LCM, of two numbers is the smallest number which is a multiple of both numbers. LCM(12,18)=36. Finally, two numbers are said to be *relatively prime* if their GCF is 1 — in other words, they have no common factors (except 1), so relative to each other, they appear to be prime. 12 and 18 are clearly not relatively prime, but 10 and 21 are relatively prime even though neither number individually is prime.

1. Can you find two numbers whose GCF is 10? Can you find two numbers whose LCM is 10?

2. Given two whole numbers a and b, can you find a relationship among their product ab, GCF(a, b), and LCM(a, b)? (Hint: Try making a table with some examples.)

3. Can you find two numbers, each larger than 100, which are relatively prime but not individually prime?

4. These definitions can be extended to larger sets of numbers. What would GCF(4,8,10) be? GCF(4,8,11)? What about LCM(4,8,10)? LCM(4,8,11)? Is either of these sets of numbers relatively prime?

5. * Jupiter orbits the sun once every 12 years. Saturn orbits the sun once every 30 years. They were in the same position in the sky in 1978. When will they next be in the same position?

6. * The U.S. Census Clock in Washington, D.C. has signs with flashing lights to indicate gains and losses in population. Here are the time periods of these flashes in seconds: birth, 10; death, 16; immigrant, 81; emigrant, 900. In other words, every 10 seconds there is a birth, and so on.

   (a) If the immigrant and emigrant signs lit up at the same time, how long will it be before they light up simultaneously again?

   (b) What is the increase in population during a one-hour period?

Divisibility Tests

1. Write down any simple tests you know to check divisibility by one of the following numbers. Try to find tests for those you don't know, but it's all right to leave a few blank. Then try to write down an explanation for why each test works.

   To check divisibility by...

   - **2:**
   - **3:**
   - **4:**
   - **5:**
   - **6:**
   - **7:**
   - **8:**
   - **9:**
   - **10:**
   - **11:**

2. Which divisibility tests are similar to the test for divisibility by 10? Why?

3. Using your answer to the previous question, what other divisibility tests *not listed above* should be similar to the test for divisibility by 4? Why?

4. Which divisibility tests are similar to the test for divisibility by 9? Why?

Odd and Even in Base Five

In our base ten place value system, we can use a quick and easy method to determine whether a quantity is even or odd without having to divide the quantity by 2 to see what the remainder is. Of course, that "quick and easy method" is simply to look at the units digit. If the units digit is even, the quantity is even; if the units digit is odd, the quantity is odd. For example, 264,967 is odd and 1,555,908 is even.

But what about place value systems that are based on numbers other than ten? Take five as a base, for example. Is there a quick and easy method of determining whether a quantity expressed in base five is even or odd without having to perform a division calculation and without converting the quantity to base ten and using our base ten algorithm? Does our base ten procedure, where we just have to observe the evenness or oddness of the units digit, work in base five? Or must we develop a new procedure?

Here are a few hints as to how you might organize your search for such an algorithm and also what you should address in writing your solution. First, using quantities expressed in base five, make a list of even numbers and a list of odd numbers (up through 3 digits). Check to see if our base ten algorithm (just look at the units digit) works for the base five number system. If not, then begin to look for a pattern and experiment with different strategies for determining whether a quantity is even or odd. Once, you think you have found an algorithm that works, make sure that you test the algorithm on a reasonable set of quantities. Then, clearly describe and demonstrate this algorithm for your reader. Finally, write a justification for your algorithm (i.e., prove that your algorithm will work for **all** numbers expressed in base five).

The Lockers Problem

The lockers at Martin Luther King Middle School are numbered from 1 to 100. (It's a small school.) On the first day of summer, the janitor opens all of the lockers. The next day, the janitor comes through and closes every second locker (that is, those numbered 2, 4, 6, ...). On the third day, the janitor goes through and changes every third locker: if it is open s/he closes it and if it is closed s/he opens it. The next day, the janitor changes every fourth locker, and so on. On the 100th day of summer, just before school starts again, the janitor changes every 100th locker (i.e., just the last locker). Which lockers are open at the end of all of this opening and closing of locker doors?

The Stamps Problem

1. Suppose you have an unlimited supply of 4 cent stamps and 9 cent stamps. What amounts *CAN'T* you make with these stamps? (Hint: organize your data in a useful way, perhaps in a table or chart, structured using the stamp denominations.)

2. What amounts can't you make with a supply of 9 cent stamps and 21 cent stamps?

3. Can you make any generalizations about what can and cannot be made with a supply of a cent stamps and b cent stamps?

Here are some things to think about in order to answer the question:

- What accounts for the qualitative difference in the answers to questions 1 and 2?
- When, in general, will the solution be like that in question 1? When like question 2?
- What is the *last* number you CAN'T make? (Adapt the question as necessary.)

How Many Factors?

1. Use the table you made at the beginning of this unit with the numbers from 1 to 100, to classify each number by how many factors it has. Make a new set of tables, one for each number of factors. Each line in the new table should include the number (from 1 to 100) and its prime factorization, nothing else.

2. Using your new set of tables,
   (a) What do numbers with exactly one factor look like?
   (b) Exactly two factors?
   (c) Exactly three factors?
   (d) Exactly four factors? (careful—there are two possibilities here)

72	2^0	2^1	2^2	2^3
3^0				
3^1				
3^2				

3. Note $72 = 2^3 \cdot 3^2$. Fill out the table at left with the products of the headers of each row and column.
   (a) How are the numbers you have placed in the table related to 72? Explain.
   (b) How many factors does 72 have?
   (c) Make similar tables for the numbers 54 and 225.

4. Fill in the table below, using your earlier results and trying to extend any patterns you see.

Numbers with this many factors...	look like this....
1	
2	
3	
4	possibility 1:
	possibility 2:
5	
6	possibility 1:
	possibility 2:
7	
8	possibility 1:
	possibility 2:
	possibility 3:
9	possibility 1:
	possibility 2:
10	possibility 1:
	possibility 2:

5. What patterns do you see in the table in the previous question?

6. What can you say in general about the number of factors of some power of a single prime?

7. Why are there multiple possibilities for some numbers of factors, and only one possibility for others? What determines how many different prime factorization structures lead to the same number of factors?

8. What is the relationship between how many factors a number has and the powers to which primes appear in its prime factorization?

9. What do numbers with exactly 12 factors look like?

10. What is the first (smallest) number with exactly eighteen factors?

The Three Stooges Stamp Problem

Moe, Larry and Curly decide to open a post office. Each takes a sheet of a different denomination of stamp (each of which costs less than 25c). Moe tells Curly, “Larry and I are going to make all the combinations of our two stamps we can, and you put them in order and tell us what we’re making.”

Curly watches Moe and Larry make different amounts and comments, “Hey, you’re missing some. After the first few, now you’re making every third amount. See? 42, 45, 48, ...”

“You numbskull!” yells Moe. “You must not be doing it right. Come here and work with me.” He tells Larry to go out and take Curly’s place.

Larry watches Moe and Curly make different amounts with their stamps. He reports, “Hey, Moe, he’s right — you two are missing some amounts. But it’s every *fourth* amount you’re making. See? 40, 44, 48, ...”

“You’re both numbskulls!” yells Moe. “Here, you two do it, and I ’ll watch.”

Moe watches Larry and Curly make different amounts with their two stamps. After a few moments, he exclaims, ”You’re making every *fifth* amount now!” He knocks their two heads together.

1. If all three observations are correct, what are the stamp denominations that Moe, Larry and Curly each have?

2. If all three stooges worked together, making every possible combination of all three types of stamp, would they eventually be able to make every amount? How do you know?

The Three Stooges Stamp Problem Redux

There has been some trouble at the local post office, so you are called in as consultant. There are 3 employees, but only 2 of them work at a time. (Work schedule: Mon. & Tues.: Moe & Larry, Wed. & Thu.: Moe & Curly, Fri. & Sat.: Larry & Curly)

Moe is only licensed to sell 15¢ stamps. Larry is only licensed to sell 14¢ stamps. Curly, the new employee, is applying for a license to sell a single denomination of stamp, some amount between 10¢ and 20¢ (inclusive) different than those sold by Moe and Larry.

1. For which of Curly’s choices will it ALWAYS be true that on days when he works, any amount past a certain point can be made?

2. For which of Curly’s choices will it NEVER be true that on days when he works, any amount past a certain point can be made?

Taxman

"Taxman" is a game commonly used to promote factoring skills. (You can find versions of it on the Internet which will run on graphing calculators.) Below are the rules. Read the rules and play a few games, recording choices and scores each time. Then discuss in your group what the optimal strategy should be, to win (or maximize your score). Write down your strategy and be ready to explain why it is good. Write down the score it will produce using a grid of 1–36.

Taxman: The Rules. This is a one-player game in which you try to beat the "Taxman", i.e., at the end of the game you want to end up with more points than the Taxman. You begin with a grid of numbers from 1 to 36 (actually you can use any size of grid, but this is a good size for now).

1. Choose any number on the grid which has at least one factor (other than itself) that has not been crossed off. You receive that number of points. Cross your number off the grid.

2. The Taxman now receives the points for all factors of your number which have not already been crossed off (for example, if you choose 8, and the number 1 is already crossed off but 2 and 4 are not, then the Taxman receives 2+4=6 points). Cross off those factors. Note that since you had to choose a number with at least one proper factor not crossed off, the Taxman will always get some points.

3. Repeat steps 1 and 2 until there are no more numbers with any untaken factors. At this point the Taxman gets points for all the remaining numbers. Add point totals for yourself and for the Taxman. You win if you have a higher total.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Reflections

Reflection 1. What is mathematics?

Here are some questions to consider. Don't try to answer all of them in your reflection, but read through them all at least once to get some ideas. Your reflection should be at least a page in length. **NOTE:** Don't just say, "Math is important, math is in everything, we use math every day" – this does little to help explain what math *is*, or to address any of these prompts. After you've written your paper, if you go back and blank out the word "math" everywhere it appears, would a reader still know what you were talking about?

- Is math invented or discovered?

  Do things already have a certain structure, or do we impose structure on them?

  Would math be different if it had been done (or dominated) by a different set of people (or a different culture)?
- Is new mathematics still being done/created/discovered? (If so, is any of it really useful?)
- What do mathematicians do?
- What do you want your students to know about it — what kind of impression do you want them to get?
- Mathematics can be involved in several daily activities, like measuring, estimating, and finances. Do any of these activities require more math than arithmetic and maybe basic algebra? Apart from scientists and engineers, do we really need to know any other math than that?
- Why do so many people have memorably bad experiences with math (as opposed to history or other subjects)? Is it something inherent in the subject matter? How would you change things if you had the power to do so?
- Is it worthwhile (or useful) to be able to apply algorithms that one doesn't really understand?
- How far should we extend our definition of math? With recognizing patterns, for instance: would we include recognizing the pattern of phases of the moon? What about recognizing things like personality types, handwriting, or æsthetics (where to hang a picture or place furniture)?
- How has technology changed the face of math? (Are computer proofs legitimate?)
- Is math fair? Are some people naturally gifted at math and others not?
- Is there anything after calculus?
- Is math value-free? Before you answer, consider the following story.

  A few years ago, it was found that inner-city kids were answering a question on bus fares incorrectly on a national assessment test. The question was something like this:

  > You can buy a one-way bus fare for fifty cents, or a bus pass that is good for a month but costs thirty dollars. If you take the bus to school and back 20 times a month, should you buy the monthly pass or the daily fares?

  The kids mostly wanted to buy the pass, even though it appears to be the more expensive option. When questioned, they said that you should buy a pass because then other members of the family can use it at night. So it's clear that in this case, there were underlying cultural assumptions which entered into the problem.

Reflection 2. Personal Dynamics in the Math Classroom.

1. Recall your best and worst experiences learning math in the past. What characteristics can you remember that made the teacher in each case an effective or ineffective teacher? (Focus on teaching characteristics, as opposed to personal characteristics.)

2. Having worked in a cooperative group for at least a short time, what are good qualities for a group member? What does it take for a small group to work well together? Is it important for all group members to have the same qualities? What are your strengths and weaknesses as a group member?

Reflection 3. Hard problems.

1. What makes a problem hard?

2. Are hard problems of value? Why or why not?

3. Are they harmful?

4. Will you assign hard problems?

5. In this age of political correctness and accommodating disabilities, should we hold different students to different levels of accountability for critical thinking?

Reflection 4. Epiphanies.

An epiphany is an event in which the essence of something is suddenly revealed to you, in a flash of recognition, like a light suddenly coming on. This course is designed to provide opportunities for epiphanies large and small, either about individual problems or about the nature of mathematics, or about what it means to do mathematics.

Describe one such moment that you have experienced in this class. Be specific: what was the trigger event? What did you understand differently afterward?

Reflection 5. Letter to the Future.

It is our hope that you have learned some things in this class that you didn't know before, beyond just the specific mathematical topics we discussed. Probably there are things you know now that you wish you had known at the beginning of the semester. For this assignment you are to write a short letter to the future students of this course, and tell them these things. Here are a few guidelines:

- Don't give answers to problems. Answers without the insights that reveal them are frustrating.
- Be specific. "This class sucks" or "This class is fun" is not helpful. "This class requires a lot of outside studying, but at first I thought it wouldn't" is.
- Consider how your image of you doing mathematics has changed.
- Consider in general what worked for you, or what helped you the most, and what didn't.
- Don't mention individuals by name (instructors included).

These letters will be kept for the next generation of students and read on the first day of class. To preserve anonymity (and allow a grade to be returned without returning the letter), put your name on a cover page only.

A Guide to Writing in Mathematics Classes
Dr. Annalisa Crannell[1]
Franklin & Marshall College

Adapted for Math 130 at UW-Madison by Brad Franklin and Christopher Kribs, 1997.

Table of Contents

§ 1. Why Should You Have To Write Papers In A Math Class?

For most of your life so far, the only kind of writing you've done in math classes has been on homeworks and tests, and for most of your life you've explained your work to people that know more mathematics than you do (that is, to your teachers). But soon, this will change.

Now that you are taking college mathematics, you will soon know more mathematics than many Americans remember. With each additional mathematics course you take, you further distance yourself from the average person on the street. You may feel like the mathematics you can do is simple and obvious, but you can be sure that other people find it bewilderingly complex. It becomes increasingly important, therefore, that you can explain what you're doing to others that might be interested: your parents, your boss, the media, your students.

Nor are mathematics and writing far-removed from one another. Professional mathematicians spend most of their time writing: communicating with colleagues, applying for grants, publishing papers, writing memos and syllabi. Writing well is extremely important to mathematicians, since poor writers have a hard time getting published, getting attention from the Deans, and obtaining funding. It is ironic but true that most mathematicians spend more time writing than they spend doing math.

But most of all, one of the simplest reasons for writing in a math class is that writing helps you to learn mathematics better. By explaining a difficult concept to other people, you end up explaining it to yourself.

Every year, we buy ten cases of paper at $35 each; and every year we sell them for about $1 million each. Writing well is very important to us.

— Bill Browning, President of Applied Mathematics, Inc.

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§ 2. How is Mathematical Writing Different from What You've Done So Far?

A good mathematical essay has a fairly standard format. We tend to start solving a problem by first explaining what the problem is, often trying to convince others that it's an interesting or worthwhile problem to solve. On your homeworks, you've usually just said, "9(a)" and then plunged ahead; but in your formal writing, you'll have to take much greater pains.

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After stating what the problem is, we usually then state the answer, even before we show how we got it. Sometimes we even state the answer right along with the problem. It's uncommon, although not so uncommon as to be exceptional, to read a math paper in which the answer is left for the very end. Explaining the solution and then the answer is usually reserved for cases where the solution technique is even more interesting than the answer, or when the writers want to leave the readers in suspense. But if the solution is messy or boring, then it's typically best to hook the readers with the answer before they get bogged down in details.

Another difference is that when you do your homework, it is important to show exactly how you got your answer. However, when you write to a non-mathematician, sometimes it's better to show why your answer works, with just a brief explanation as to how you got it. For example, compare:

Homework Mathematics:

To solve for x when $3x^2 - 21x + 30 = 0$, we use the quadratic formula:

$$\begin{aligned} x &= \frac{21 \pm \sqrt{21^2 - 4 \cdot 3 \cdot 30}}{2 \cdot 3} \\ x &= \frac{21 \pm \sqrt{441 - 360}}{6} \\ x &= \frac{21 \pm \sqrt{441 - 360}}{6} \\ x &= \frac{30}{6} \text{ or } \frac{12}{6} \\ x &= 5 \text{ or } 2 \end{aligned}$$

and so either $x = 5$ or $x = 2$.

More Formal Mathematics:

To solve for x when $3x^2 - 21x + 30 = 0$, we used the quadratic formula and found that either $x = 5$ or $x = 2$. It's easy to see that these are the right answers, because

$$(3 \times 5^2) - (21 \times 5) + 30 = 75 - 105 + 30 = 0,$$

and also

$$(3 \times 2^2) - (21 \times 2) + 30 = 12 - 42 + 30 = 0.$$

The difference is that, in the first example, you're trying to convince someone who knows a lot of math that you, too, know what you're doing (and if you don't, to get partial credit). In the second example, you're trying to show someone who may or may not be good at math that you got the right answer.

Math is difficult enough that the writing around it should be simple. 'Beautiful' math papers are the ones that are the easiest to read: clear explanations, uncluttered expositions on the page, well-organized presentation. For that reason, mathematical writing is not a creative endeavor the same way that, say, poetry is: you shouldn't be spending a lot of time looking for the perfect word, but rather should be developing the most clear exposition. Unlike humanities students, mathematicians don't have to worry about over-using 'trite' phrases in mathematics. In fact, at the end of this booklet are a list of trite but useful phrases that you may want to use in your papers, either in this class or in the future.

This guide, together with your checklist, should serve as a reference while you write. If you can master these basic areas, your writing may not be spectacular, but it should be clear and easy to read — which is the goal of mathematical writing, after all.

§ 3. A Checklist

Use this checklist as a guide for yourself while writing a problem report. These are the sorts of things on which the write-ups are graded.

Does this paper:

1. clearly (re)state the problem to be solved?
2. state the answer in a complete sentence which stands on its own?
3. provide a paragraph which explains how the problem will be approached?
4. define all variables, terminology, and notation used?
5. clearly state the assumptions which underlie the formulas and theorems?
6. explain how each formula or theorem is derived, or where it can be found?
7. clearly label diagrams, tables, graphs, or other visual representations of the math?
8. give a precise and well-organized explanation of how the answer was found?
9. give acknowledgment where it is due?
10. use correct spelling, grammar, and punctuation?
11. contain correct mathematics?
12. solve the question that was originally asked?

Here is a more detailed explanation of the criteria.

1. Clearly restate the problem to be solved.

Do not assume that the reader knows what you're talking about. (The person you're writing to might be out on vacation, for example, or have a weak memory). You don't have to restate every detail, but you should explain enough so that someone who's never seen the assignment can read your paper and understand what's going on, without any further explanation from you. Outline the problem carefully.

2. State the answer in a complete sentence or paragraph.

If you can avoid variables in your answer, do so; otherwise, remind the reader what they stand for. If your answer is at the end of the paper and you've made any significant assumptions, restate them, too. Do not assume that the reader has actually read every word and remembers it all (do you?).

3. Provide a paragraph which explains how you and others approached the problem.

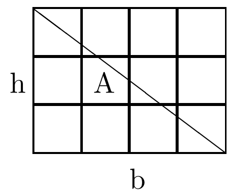
It's not polite to plunge into mathematics without first warning your reader. Carefully outline the steps you're going to take, giving some explanation of why you're taking that approach. It's nice to refer back to this paragraph once you're deep in the thick of your calculations.

This is where you give some details about what happened in small and large group. Don't go overboard — just highlight a couple of key moments and ideas (even ones that didn't work). You may want to explain how you chose to interpret the problem if issues arose in this.

4. Define all variables used.

(a) Even if you label your diagram (and you should), you should still explain in words what your variables are.

(b) If there's a quantity you use only a few times, see if you can get away with not assigning it a variable. As examples:

 Figure 1: *Diagram of the triangle* (each square is 1" by 1")	We see that the area of the triangle will be one-half of the product of its height and base — that is, the area of the triangle is $(1/2) \times 3 \times 4 = 6$ square inches.	√
	We see that $A = (1/2) \times b \times h$, where A stands for the area of the triangle, b stands for the base of the triangle, and h stands for the height of the triangle, and so $A = (1/2) \times 3 \times 4 = 6$ square inches.	×

Elementary physics tells us that the velocity of a falling body is proportional to the amount of time it has already spent falling. Therefore, the longer it falls, the faster it goes.	√
Elementary physics tells us that $v_t = g(t - t_0)$, where v_t is the velocity of the falling object at time t, g is gravity, and t_0 is the time at which the object is released. Therefore as t increases, so does v_t: i.e., as time increases, so does velocity.	×

I hope that you'll agree that the first example of each pair is much easier to read.

(c) The more specific you are, the better. State the units of measurement. When you can use words like "of", "from", "above", etc., do so. For example:

We get the equation $d = rt$, where d is the distance, r is the rate, and t is the time.	×
We get the equation $d = rt$, where d is the distance from Sam's car to her home (in miles), r is the speed at which she's traveling (measured in miles per hour), and t is the number of hours she's been on the road.	√

Avoid words like "position" (height above ground? sitting down? political situation?) and "time" (5 o'clock? January? 3 minutes since the experiment started?). Never mind that your instructor uses these words freely; you can too when you get a Ph.D.

(d) Variables in text are italicized to tell them apart from regular letters.

5. Clearly state the assumptions which underlie the formulas.

For example, what physical assumptions do you have to make? (No friction, no air resistance? That something is lying on its side, or far away from everything else?) Sometimes things are so straightforward that there are no assumptions, but not often.

6. Explain how each formula is derived, or where it can be found.

Don't pull formulas out of a hat, and don't use variables which you don't define. Either derive the formula yourself in the paper, or explain exactly where you found it, so other people can find it, too. Put important or long formulas on a line of their own, and then center them; it makes them much easier to read:

The total number of infected cells in a honeycomb with n layers is $$1 + 2 + ... + n = n(n+1)/2.$$ Therefore, there are $100(101)/2 = 5,050$ infected cells in a honeycomb with 100 layers.	√
The total number of infected cells in a honeycomb with n layers is $1+2+\ldots+n = n(n+1)/2$. Therefore, there are $100(101)/2 = 5,050$ infected cells in a honeycomb with 100 layers.	×

Sufficiently advanced copies of Microsoft Word have an equation editor: pull down the *Insert* menu, select *Object*, and then *Equation*. If you don't have an equation editor, you may either try formulas with tabs and fancy fooling, you may use Excel for matrices if you know how, or you may wish to write the mathematics in by hand. All of these are fine options.

A caution: Mathematical formulas are hard to do on a standard word processor. If you've got something that's really complicated, feel free to leave space and then write it out by hand later– but then don't forget to do so! (Like I often do).

7. Clearly label diagrams, tables, graphs, or other visual representations of the math (if these are indeed used).

In math, even more than in literature, a picture is worth a thousand words, especially if it's well labeled. Label all axes, with words, if you use a graph. Give diagrams a title describing what they represent. It should be clear from the picture what any variables in the diagram should represent. The whole idea is to make everything as clear and self-explanatory as possible. If you decide to draw pictures in MacDraw or SuperPaint (which you can then copy to Write Now, Microsoft Word, or MacWrite), you will (a) have a lot of fun, and (b) waste a lot of time. If you're going to draw diagrams on the computer, I recommend doing so in the early stages of the project, and then saving the document so you can change it later — because you'll have to change it later, and you can't do it once it's in Write Now or whatever.

8. Give a precise and well-organized explanation of how the answer is found.

This is the single most important step.

9. Give acknowledgment where it is due.

Plagiarism is almost certainly the greatest sin in academia — some fiction writers make plagiarism a motive for murder. It's extremely important to acknowldge where your inspiration, your proofreading, and your support came from. In particular, you should cite: any book you look at, any computational or graphical software which helped you understand or solve the problem, any student you talk to (whether in this class or not), any professor you talk to (including and especially me, because I'll catch you if you leave me out). The more specific you are, the better.

10. In this paper, are the spelling, grammar, and punctuation correct?

(a) It may surprise you that it is on spelling and grammar that people tend to lose most of their points on their mathematics papers. Please spell-check and proofread your work for grammar mistakes. Better yet, ask a friend to read your paper. Mathematicians are generally not petty, but neither are we amused by sloppy or careless writing.

(b) Mathematical formulas are like clauses or sentences: they need proper punctuation, too. Put periods at the end of a computation if the computation ends the sentence; use commas if it doesn't. An example follows.

If Dr. Crannell's caffeine level varies proportionally with time, we see that $$C_t = kt,$$ where C_t is her caffeine level t minutes after 7:35 a.m., and k is a constant of proportionality. We can solve to show $k = 202$, and therefore her caffeine level by 11:02 (t=207) is $$\begin{aligned} C_{207} &= 202 \times 207, \\ &= 41,814. \end{aligned}$$ In other words, she's mightily buzzed.	√

(c) Do not confuse mathematical symbols for English words (= and # are especially common examples of this). The symbol "=" is used only in mathematical formulas — not in sentences:

We let V stand for the volume of a single mug and n represent the number of mugs. Then the formula for the total amount of root beer we can pour, R, is $R = nV$.	$\surd$
We let V = volume of a single mug and n = the # of mugs. Then the formula for the total amount of root beer $R = nV$.	$\times$
We let V stand for the volume of the mug and n represent the number of mugs. Then the formula for the total amount of root beer we can pour, R, is R is nV.	$\times$

(d) Do, however, use equal signs when you state formulas or equations, because mathematical sentences need subjects and verbs, too.

Then the formula for the total amount of root beer we can pour is $R = nV$.	$\surd$
Then the formula for the total amount of root beer we can pour is nV.	$\times$

11. In this paper, is the mathematics correct?

This is self-explanatory.

12. In this paper, did the writer solve the question that was originally asked?

So is this.

§ 4. Good Phrases to Use in Math Papers:

• Therefore (*also*: so, hence, accordingly, thus, it follows that, we see that, from this we get, then)
• I am assuming that (*also*: assuming, where, M stands for; *in more formal mathematics*: let, given, M represents)
• show (*also*: demonstrate, prove, explain why, find)
• (see the formula above). (*also*: (see *), this tells us that ...)
• if (*also*: whenever, provided that, when)
• notice that (*also*: note that, notice, recall)
• since (*also*: because)
• Cite sources, e.g., This formula can be found on page 234 of *Discovering Algebra* (1999), by Levine and Rosenstein.

For more information, see Crannell, Annalisa, "How to Grade 300 Mathematical Essays and Survive to tell the Tale," PRIMUS **4**, 3 (1994): 193-201.

Problem Report Tips

- Speak out during large group discussion to get other groups' ideas. Make them explain their ideas to you, so you can explain them clearly to me in your problem report.
- Address every question: even if you do not know the answer, at least acknowledge that and perhaps guess at it or suggest a strategy for finding it.
- Problem reports will ideally all have one or two key sentences which hit upon exactly why the answer you provide is true. For example, in a write-up analyzing the game Tic Tac Toe, a sentence or two which says something like, "Go first and choose the center. On your next two turns choose two adjacent squares (a corner and the center of a side), so that you have two ways to win — diagonally and straight across or down — and your opponent can do no better than a tie," hits the nail on the head.
- Sometimes you may not be able to explain the complete solution. In this case, try to pinpoint the difficulties you are having, and even try to solve a simpler version of the problem. You get partial credit for this, and it may even give you insight that helps you make the leap to solve the original problem. If there is a problem with your solution, point it out clearly, and, if possible, give special conditions under which your solution still works.
- It's better to take a stab at an answer (i.e., to guess intelligently) than to throw up your hands and say, "I don't know." Try to eke out as much of an answer (again, maybe only a partial answer) as possible, and say why you believe it.
- If something relevant comes up in small or large group that is interesting, you might want to tell about it in your report, even if it is not necessary for the solution (however, it should at least be relevant). Report ideas that did not turn out to be correct as well as those which did, if they help you to understand the pitfalls in solving a tricky problem.
- Write a conclusion that actually draws a conclusion, rather than simply rewriting your introduction. The overall flow should be "here's what I'm going to tell you about, here I'm telling you about it, and this is what we can come away with", not "here's what I'm going to tell you, here I'm telling you about it, and here's what I just told you".
- Use paragraphs — one for each idea. It's not a rule, but you usually should have more than one paragraph per side of a page.
- Is order important? There is no unique formula for writing a problem report. Use your judgement to decide the best order in which to present your report of what your group did and your argument for your solution. They may even be mixed together, if you find that useful.
- Have a friend read the report. Can s/he understand it with no help from you?
- A good problem report is exhaustive yet not verbose.
- Make it clear whether a claim is based on examples or proof (example $\neq$ proof).
- Tell the reader what you're doing — don't just give the puzzle pieces, put them together.
- Don't just transcribe class notes verbatim: reorganize and re-synthesize.